Smooth Complete Intersections with Positive-Definite Intersection Form

by

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Abstract

We classify the smooth complete intersections with positive-definite intersection form on their middle cohomology. There are two families. The first family are quadric hypersurfaces in $\mathbb{P}^{4k+1}$ with k a positive integer. The middle cohomology is always of rank two and the intersection lattice corresponds to the identity matrix. The second family are complete intersections of two quadrics in $\mathbb{P}^{4k+2}$ (k a positive integer). Here the intersection lattices are the $\Gamma_{4(k+1)}$ lattices; in particular, the intersection lattice of a smooth complete intersection of two quadrics in $\mathbb{P}^6$ is the famous E_8 lattice.

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Chapter 1

Introduction

Let X be a smooth projective variety of dimension n over the complex numbers. It is immediate from the Jacobian criterion and the holomorphic implicit function theorem that X has the structure of a compact complex manifold. From this point of view, X has a topology distinct from its Zariski topology—the so-called classical or analytic topology—which is much better suited for applications of the classical machinery of algebraic topology than the Zariski topology.

In this document, we are interested in the intersection form on the middle cohomology of X (with X viewed with its analytic topology). We begin with its definition. Let $H^k(X,\mathbb{Z})$ denote the k-th singular cohomology group of X, let $H^k(X,\mathbb{Z})_{\mathrm{tors}}$ denote the subgroup of torsion elements in $H^k(X,\mathbb{Z})$ and set $H^k(X) \coloneqq H^k(X,\mathbb{Z})/H^k(X,\mathbb{Z})_{\mathrm{tors}}$. We have

Theorem (Poincaré duality)**.** *For all $0 \le k \le n$, the cup product pairing*

$$H^{n-k}(X) \otimes_{\mathbb{Z}} H^{n+k}(X) \xrightarrow{\cup} H^{2n}(X) \cong \mathbb{Z}$$

is a perfect pairing.

In the case $k = 0$, Poincaré duality gives a unimodular bilinear form on $H^n(X)$. We call this form the *intersection form* or *cup product form* of X.

The intersection form is an interesting invariant of a compact oriented topological manifold. One striking result (due to Freedman) is that to each symmetric unimodular form B there corresponds either a unique homeomorphism type of topological fourfolds (if B is even) or exactly two distinct homeomorphism types (if B is odd) having B as the intersection form on their middle cohomology. Hence, the homeomorphism type of a topological fourfold is essentially determined by its intersection form.

Returning to smooth projective varieties, we begin with the following questions: given X, what is the intersection form on X? Conversely, given a unimodular bilinear form B, does there exist X with B as the intersection form on $H^n(X)$? These questions are hard in the stated generality and we immediately begin to refine them. Observing that the intersection form is symmetric when the (complex) dimension of X is even and alternating when the dimension of X is odd by the identity $\alpha \cup \beta = (-1)^n \beta \cup \alpha$ for $\alpha, \beta \in H^n(X)$, we are led to recall the classification of alternating and symmetric unimodular bilinear forms over the integers.

The case of alternating unimodular forms is not very interesting—up to isomorphism, there is a single alternating unimodular form on $\mathbb{Z}^r$ if r is even and none if r is odd.

The case of symmetric unimodular forms further splits into definite and indefinite subcases. In the indefinite case, we have a complete classification: a form is completely determined by its rank, signature and type [1]. In particular, for a given rank r, the number of distinct indefinite unimodular bilinear forms on $\mathbb{Z}^r$ is bounded above by $2r$. These forms can be described as follows:

Type I (odd) case: Each form may be brought by a change of basis to one represented by a diagonal matrix, with the diagonal entries equal to either $+1$ or -1 (and the number of entries of each type is equal for any choice of basis bringing the Gram matrix to this form). Letting b_+ and b_- denote the number of $+1$'s and -1's on the diagonal, respectively, the signature of the form is equal to $b_+ - b_-$, while the rank is equal to $b_+ + b_-$. It is clear that knowing the signature and the rank determines

[1] Please see section 2.1 for definitions

the form.

Type II (even) case: It may be shown that each form may be brought by a change of basis to $U^{\oplus a} \bigoplus E_8^{\oplus b}$, where a is a nonnegative integer, b is an arbitrary integer, and U and $E_8 = \Gamma_8$ are lattices described in section 2.1 ([13] Theorem V.5, pg. 54). The signatures of the two lattices are $\sigma(U) = (1+, 1-)$ and $\sigma(E_8) = (8+, 0-)$, so that the signature of the form is equal to $8b$. The rank is equal to $2a + 8|b|$, so again we see that the rank and signature determine the form.

In the definite case, we again split into positive-definite and negative-definite subcases. However, there is no loss of generality in considering only one of the subcases (scaling a form of one class by -1 gives a form of the other class). We note that, with X as before and $[H] \in H^1(X, \mathbb{Z})$ the class of a hyperplane section of X, we have $[H]^n \cdot [H]^n = \deg(X) > 0$. Hence, the intersection form of a smooth projective variety is never negative-definite and we focus on describing the positive-definite case.

Unfortunately, no complete classification of positive-definite symmetric unimodular forms is known (to the author). It is known, however, that the number of distinct isomorphism classes grows very rapidly past rank 24. In particular, recalling that the rank of a form of type II must be divisible by 8, the number of distinct type II positive-definite unimodular forms is given in the following table:

Rank	8	16	24	32	40
Number of distinct forms	1	2	24	$\geq 10^7$	$\geq 10^{51}$

([12] pg. 28). The type I case exhibits even more rapid growth ([9] pg. 18). This behaviour is quite different from the indefinite case, where the number of distinct indefinite forms grows linearly with the rank. Despite this, as a heuristic principle smooth projective varieties with positive-definite intersection form are 'rare'. For example, the only Grassmannians with nonempty middle cohomology and positive-definite intersection form are the two families $\mathbb{G}(1, n)$ and $\mathbb{G}(n-2, n)$ (as demonstrated in chapter 3). The following result, proved as corollary 4.1 below, gives another illustration of our principle:

Theorem. *With notation as above, if X has a positive-definite intersection form, then the rank of the Néron-Severi group of X is one.*

Many of the basic constructions one might want to use to produce examples of smooth projective varieties with positive-definite intersection form strictly increase the Picard rank. For example, this is true for taking products or blowing-up along a proper subvariety. So the above theorem fits the imprecise principle that smooth projective varieties X with positive-definite intersection form are rare.

In this document, we study the intersection form on the middle cohomology of the varieties in the following class:

Definition 1.1. A variety $X \subset \mathbb{P}^{n+r}$ of dimension n is called a *complete intersection of type* $(d_1, \ldots, d_r)$ (here $d_1 \leq \cdots \leq d_r$) if its ideal is generated by exactly r homogeneous polynomials $f_1, \ldots, f_r$ in $\mathbb{P}^{n+r}$ of degrees $d_1, \ldots, d_r$, respectively. We often denote a complete interesection of type $\mathbf{d} \coloneqq (d_1, \ldots, d_r)$ by $X(\mathbf{d})$.

Our main result is a complete classification of the smooth complete intersections with positive-definite intersection form—the following statement combines theorems 4.2, 5.1 and 5.4.

Theorem. *The smooth complete intersections with positive-definite intersection form are exactly:*

1. *The smooth quadric hypersurfaces in $\mathbb{P}^{4k+1}$ (k a positive integer), with intersection form*

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and

2. *The smooth complete intersections of two quadrics in $\mathbb{P}^{4k+2}$ (k positive), with intersection form $\Gamma_{4(k+1)}$.*[2]

[2]Please see section 2.1 for a description of the lattices Γ_{4k}.

In particular, in line with the above, complete intersections with positive-definite intersection form are ‘rare’ among all complete intersections.

In the rest of this document, we pursue a more-or-less direct path toward a proof of the above theorem. Chapter 2 is devoted to preliminaries, in particular describing the family of lattices Γ_{4n} appearing in the statement of the theorem above and describing the excess intersection formula. Explicit descriptions of the integral cohomology rings of the projective spaces $\mathbb{P}^n$ and Grassmannians $\mathbb{G}(k,n)$ also appear in Chapter 2, making it possible to analyze the question of which of the spaces $\mathbb{P}^n$ and $\mathbb{G}(k,n)$ have a positive-definite intersection form in Chapter 3. In Chapter 4, we classify the types of complete intersections with positive-definite form, using the Hodge-Riemann bilinear relations and an inequality bounding the type of a complete intersection in terms of the coniveau of its middle cohomology. In Chapter 5, we identify the lattices that appear by finding explicit generating cycles in the middle cohomology.

We end this introduction by listing a few conventions. Throughout the following, we shall be working over the complex numbers. In particular, dimensions are complex dimensions unless otherwise stated. We often denote the dimension of a smooth projective X by n. Finally, unless otherwise stated tensor products are to be taken over the integers.

Chapter 2

Preliminaries

2.1 Lattices

In this section, we recall some of the basic invariants of lattices mentioned in the introduction and describe the family of lattices Γ_{4k}. Our main references are [12] chapters 1 and 2 and [13] chapter V.

Let A be a commutative ring with unit 1 and let M be an A-module.

Definition 2.1. An A-bilinear map $M \times M \longrightarrow A$ is called a *bilinear form* on M; we often denote the image of (m, n) by $m \cdot n$. A pair $(M, \cdot)$, where M is a left A-module and $\cdot$ a bilinear form on M is called a *bilinear form module.* A morphism of bilinear form modules $(M, \cdot_M)$ and $(N, \cdot_N)$ is an A-module map $\phi : M \longrightarrow N$ such that $\phi(m) \cdot_N \phi(m') = m \cdot_M m'$ for all $m, m' \in M$.

Definition 2.2. A bilinear form $\cdot$ on an A-module M is called *symmetric* if $m \cdot n = n \cdot m$ for all $m, n \in M$ and *alternating* if $m \cdot n = -n \cdot m$ for all $m, n \in M$.

We shall be particularly interested in the following case.

Definition 2.3. A bilinear form module $L = (V, \cdot)$ where V is a free abelian group of finite rank and $\cdot$ is a symmetric bilinear form on V is called a *lattice.* A morphism of lattices L and L' is a morphism of L and L' as bilinear form modules.

We at times abuse notation by not distinguishing between $L = (V, \cdot)$ and V. Thus the *rank* of L is by definition the rank of V and *basis* of L is by definition a basis of V (other similar conventions are left implicit).

Our main example of a lattice will be the *intersection* or *cup product lattice* $(H^n(X), \cup)$, where X is a smooth projective even-dimensional variety.

Roughly speaking, it is often possible to check certain properties of the lattice by checking associated properties of the following matrix.

Definition 2.4. Let $L = (V, \cdot)$ be a lattice of rank r and let $\{v_1, \ldots, v_r\}$ be a choice of basis of L. Then the matrix $(v_i \cdot v_j)_{i,j=1}^r$ is called the *Gram matrix* of L.

We shall frequently make the conceptual abuse of identifying a lattice with its Gram matrix. In these cases, we shall be careful to be explicit about the choice of basis.

We check how the Gram matrix of L changes under change of basis. Let $\mathrm{GL}(r, \mathbb{Z})$ be the group of invertible $r \times r$ matrices over $\mathbb{Z}$. For any two choices of basis $B = \{e_1, \ldots, e_r\}$ and $B' = \{f_1, \ldots, f_r\}$ of V, there exists a matrix $M \in \mathrm{GL}(r, \mathbb{Z})$ such that

$$\begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_r \end{pmatrix} = M \begin{pmatrix} e_1 \\ e_2 \\ \vdots \\ e_r \end{pmatrix}.$$

Let G and G' be Gram matrices of L with choice of basis B and B', respectively. Then a simple calculation shows that

$$G' = MGM^t, \tag{2.1}$$

where M^t is the transpose of M. In particular, $\det(G)$ is independent of the choice of basis for V, motivating the definition of our first invariant

Definition 2.5. The integer $d(L) := \det(G)$, where G is the Gram matrix of L with respect to a choice of basis, is called the *discriminant* of L. The lattice L is called *nondegenerate* if $d(L) \neq 0$ and *unimodular* if $d(L) = \pm 1$.

We have the useful

Lemma 2.1. *Let $L = (V, \cdot)$ be a nondegenerate lattice and $M = (W, \cdot|_W)$ is a sublattice of L of full rank. Then we have*

$$d(L)(L : M)^2 = d(M),$$

where $(L : M)$ denotes the index (the number of left-cosets) of W in V.

Corollary 2.1. *Let $L = (V, \cdot)$ be a nondegenerate lattice and $M = (W, \cdot|_W)$ be a unimodular full rank sublattice of L. Then L is unimodular and $M = L$.*

The proof of the corollary is short enough that we record it here

Proof. By lemma 2.1, we have $(L : M)^2 = \pm d(L)^{-1}$. Since the left side is a positive integer and $|d(L)| \geq 1$, we have $|d(L)| = 1$. Then $(L : M) = 1$, so $L = M$. $\square$

We continue with description of the basic lattice invariants. To each lattice L, we can associate the quadratic form $Q_L : v \mapsto v \cdot v$. With a choice of basis $B = \{e_1, \dots, e_r\}$ of L, Q_L looks as follows: let $G = (g_{ij})_{i,j=1}^r$ be the Gram matrix of L (with respect to B) and let $v = \sum_i v_i e_i$ for integers v_i. Then

$$Q_L(v) = v \cdot v = \sum_{i,j=1}^r g_{ij} v_i v_j = \sum_{i=1}^r g_{ii} v_i^2 + 2 \sum_{i<j} g_{ij} v_i v_j. \tag{2.2}$$

The rest of the invariants of L are defined in terms of the quadratic form Q_L.

Definition 2.6. A quadratic form Q is called *definite* if $Q(v)$ has the same sign for all $v \in V$. If the form Q is definite, it is called *positive-definite* if $Q(v) > 0$ for all $v \in V$ and *negative-definite* if $Q(v) < 0$ for all $v \in V$. Otherwise Q is called *indefinite*. A lattice L is called *positive-definite*, *negative-definite* or *indefinite* if the quadratic form associated to L is.

Definition 2.7. A lattice $L = (V, \cdot)$ is called *even* (*Type II*) if the associated quadratic form takes on even values for all $v \in V$, otherwise L is called *odd* (*Type I*).

Let $V_{\mathbb{R}} := V \otimes \mathbb{R}$ be the $\mathbb{R}$-vector space obtained by extending scalars. Extend the bilinear form $\cdot$ to a bilinear form $\cdot_{\mathbb{R}}$ on $V_{\mathbb{R}}$ and let $L \otimes \mathbb{R} := (V \otimes \mathbb{R}, \cdot_{\mathbb{R}})$. It is well-known from linear algebra that any quadratic form over $\mathbb{R}$ may be brought to the form

$$Q : (v_1, \dots, v_n) \in V \mapsto \epsilon_1 v_1^2 + \cdots + \epsilon_n v_n^2, \tag{2.3}$$

where $\epsilon_i = \pm 1$ or $\epsilon_i = 0$ for each $1 \leq i \leq n$. Moreover, the number of indices for which ϵ_i = 1, -1 and 0 is independent of the choice of basis of V for which Q is in diagonal form (2.3) (so that these numbers depend only on Q). We then have the definitions

Definition 2.8. The *signature* $\sigma(Q)$ of a quadratic form Q is defined to be the tuple of positive integers (b_+, b_-), where b_+ is the number of indices for which $\epsilon_i = 1$ and b_- the number of indices for which $\epsilon_i = -1$ in a diagonal representation (2.3) of Q. The *index* of the quadratic form Q of signature (b_+, b_-) is defined to be the integer $\tau(L) := b_+ - b_-$. The *signature* and *index* of a lattice L are defined to be the signature and index, respectively, of the quadratic form associated to $L \otimes \mathbb{R}$.

We finish this section with an example. The following lattice came up in the classification of even indefinite unimodular forms.

Example. *Fix the standard basis* $\{(1,0),(0,1)\}$ *of* $\mathbb{Z}^2$ *and define a lattice* U *on* $\mathbb{Z}^2$ *by the Gram matrix*

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The lattice U *called the hyperbolic plane. We see that* U *is a rank two symmetric unimodular even lattice. The associated quadratic form is*

$$(a_1, a_2) \mapsto 2a_1a_2 = \left(\frac{a_1 + a_2}{\sqrt{2}}\right)^2 - \left(\frac{a_1 - a_2}{\sqrt{2}}\right)^2,$$

so we have $\sigma(U) = (1+, 1-)$ *and* $\tau(U) = 0$. *In particular,* U *is indefinite.*

2.1.1 The lattices Γ_{4k}

We now describe the family of lattices Γ_{4k} for $k \geq 1$ following [13] section V.1.4.3. As indicated in the introduction, this family will play an important role in chapter 5.

Let $r = 4k$ be a positive integer. Equip the rational vector space $\mathbb{Q}^r$ with a rational-valued bilinear form having its Gram matrix equal to the $r \times r$ identity matrix with respect to the standard basis of $\mathbb{Q}^r$. Identify $\mathbb{Z}^r$ with the free abelian subgroup of $\mathbb{Q}^r$ consisting of points of $\mathbb{Q}^r$ with integer coordinates (with respect to the standard basis).

Consider the homomorphism of abelian groups $\mathbb{Z}^r \longrightarrow \mathbb{Z}/2\mathbb{Z}$ mapping $(v_1, \ldots, v_r) \in \mathbb{Z}^r$ to $\sum v_i \mod 2$ and let K be its kernel. We have $(\mathbb{Z}^r : K) = 2$; this fact will be needed below. Let E be the free abelian subgroup of $\mathbb{Q}^r$ generated by K and the vector $e := (\frac{1}{2}, \ldots, \frac{1}{2})$. The following is a more explicit description of the points in E: $v \in E$ if and only if

$$i).\ 2v_i \in \mathbb{Z} \text{ for all } 1 \leq i \leq r, \qquad ii).\ v_i - v_j \in \mathbb{Z} \text{ for all } 1 \leq i, j \leq r, \qquad iii).\ \sum_{i=1}^{r} v_i \in 2\mathbb{Z}. \tag{2.4}$$

The above conditions can be seen, for example, by writing out v as $ae + \ell$, with $a \in \mathbb{Z}$ and $\ell \in K$. The abelian group E with restriction of the bilinear form on $\mathbb{Q}^r$ to E is our candidate for a lattice.

To check that $(E, \cdot|_E)$ is a lattice, we must check that $\cdot|_E$ is integer-valued. If v, w are arbitrary elements of E, we may be write $v = ae + m,\ w = be + \ell$ for some $a, b \in \mathbb{Z}$ and $m, \ell \in K$, so that by bilinearity

$$v \cdot w = ab(e \cdot e) + e \cdot (am + b\ell) + m \cdot \ell. \tag{2.5}$$

We have $e \cdot e = r/4 = k$ is an integer, so $ab(e \cdot e)$ is an integer. Next, since $(am + b\ell) \in E$, $e \cdot (am + b\ell)$ is an integer by condition *iii)* of (2.4). Finally, it is clear that $m \cdot \ell$ is an integer. It follows that $v \cdot w$ is an integer. Therefore, $(E, \cdot|_E)$ is a lattice. It is this step of the construction that makes it necessary to have the dimension r of $\mathbb{Q}^r$ divisible by 4.

We denote the lattice $(E, \cdot|_E)$ by Γ_{4k}. Other common notations for Γ_8 are E_8 and D_8^+.

The lattices Γ_{4k} have the following invariants:

Lemma 2.2. *Let $k \geq 1$. We have $r(\Gamma_{4k}) = 4k$, $\sigma(\Gamma_{4k}) = (4k+, 0-)$, $\tau(\Gamma_{4k}) = 4k$ and $d(\Gamma_{4k}) = 1$. In particular, Γ_{4k} is unimodular and positive-definite. Moreover, Γ_{4k} is even exactly when k is even and odd exactly when k is odd.*

Proof. **$r(\Gamma_{4k}) = 4k$:** By construction $4k = r(\mathbb{Z}^r) \leq r(\Gamma_{4k}) \leq \dim_{\mathbb{Q}}(\mathbb{Q}^r) = 4k$.

$d(\Gamma_{4k}) = 1$: We see that $2e \in K$, but $e \notin K$, so $(\Gamma_{4k} : K) = 2$. Applying lemma 2.1 to the lattice inclusions $K \subset \mathbb{Z}^r$ and $K \subset \Gamma_{4k}$, we have

$$d(\mathbb{Z}^r)(\mathbb{Z}^r : K)^2 = d(K) \quad \text{and} \quad d(\Gamma_{4k})(\Gamma_{4k} : K)^2 = d(K).$$

Since $d(\mathbb{Z}^r) = 1$ and $(\mathbb{Z}^r : K) = 2$, it follows that $d(\Gamma_{4k}) = 1$.

Since the bilinear form on Γ_{4k} is the restriction of a positive-definite bilinear form on $\mathbb{Q}^r$ to a subset of $\mathbb{Q}^r$, it is clear that the bilinear form on Γ_{4k} remains positive-definite. It follows that

$\sigma(\Gamma_{4k})$: is equal to $(4k+, 0-)$ and so

$\tau(\Gamma_{4k})$: is equal to $4k$.

Finally, we determine whether Γ_{4k} is even or odd. By (2.5), for any $v = ae + m \in E$, we have

$$v \cdot v = a^2 k + 2a(e \cdot m) + m \cdot m. \tag{2.6}$$

Now by definition of K, if $m = (m_1, \ldots, m_r) \in K$, then $\sum m_i \equiv 0 \mod 2$, hence $\left(\sum m_i\right)^2 \equiv 0 \mod 2$. But $\left(\sum m_i\right)^2 = \sum m_i^2 + 2\sum_{i<j} m_i m_j$, so $m \cdot m = \sum m_i^2 \equiv \left(\sum m_i\right)^2 = 0 \mod 2$. It follows that $v \cdot v \equiv a^2 k \mod 2$. Therefore Γ_{4k} is even exactly when k is even and odd exactly when k is odd. □

Finally, the following set of generators for Γ_{4k} will be useful in chapter 5:

Lemma 2.3. *Let $r = 4k$ be a positive integer. Then the lattice elements $\gamma_i := e_i - e_{i+1}$ with $i = 1, \dots, (r-1)$, $\gamma_r := 2e_r$ and $\gamma_{r+1} := \frac{1}{2}(e_1 + e_2 + \cdots + e_r)$ generate Γ_r.*

Proof. Let $v = (v_1, \dots, v_r) \in \Gamma_r \subset \mathbb{Q}^r$ be arbitrary. We use the conditions (2.4) characterizing the vectors v contained in Γ_r. By conditions *i)* and *ii)*, we may assume that v_i are either integers for all $1 \le i \le r$ or (odd) integral multiples of $\frac{1}{2}$ for all $1 \le i \le r$. We may reduce the second case to the first by subtracting $\frac{1}{2}(e_1 + \cdots + e_r)$ from the vector v.

Now let $\delta_1 := v_1 \in \mathbb{Z}$ (possibly equal to 0) and bring the coordinate v_1 of v to 0 by subtracting $\delta_1(e_1 - e_2)$ from v. Proceed inductively for $j = 2, \dots, r-1$, at step j bringing the j-th coordinate of $v - \sum_{i=1}^{j-1} \delta_i(e_i - e_{i+1})$ to 0 by subtracting an appropriate multiple $\delta_j \in \mathbb{Z}$ of the vector $(e_j - e_{j+1})$ from the vector $v - \sum_{i=1}^{j-1} \delta_i(e_i - e_{i+1})$. Then every coordinate of the vector $v - \sum_{i=1}^{r-1} \delta_i(e_i - e_{i+1})$ is equal to 0, except possibly the r-th coordinate. But by condition iii) of (2.4), any such vector must be an integral multiple of $2e_r$, say $\delta_r 2e_r$. Then we have

$$v = \sum_{i=1}^{r-1} \delta_i(e_i - e_{i+1}) + \delta_r 2e_r = \sum_{i=1}^{r} \delta_i \gamma_i$$

up to adding the vector $\frac{1}{2}(e_1 + \cdots + e_r) = \gamma_{r+1}$ to the right side, which shows the claim.

□

2.2 The Hodge and Lefschetz decompositions of $H^k(X, \mathbb{C})$

The singular cohomology groups $H^k(X, \mathbb{Z})$ are homeomorphic invariants of X and in particular do not depend on the complex structure of X. On the other hand, the Hodge decomposition splits $H^k(X, \mathbb{C}) = H^k(X, \mathbb{Z}) \otimes \mathbb{C}$ into a direct sum of complex vector spaces, each of which depend on the complex structure. We may hope to obtain refined information about the intersection form on X (making use of the fact that X has a naturally defined complex structure) by extending it to $\mathbb{C}$ and studying its restriction to the pieces of the Hodge decomposition. In this section, we describe the Hodge decomposition of $H^k(X, \mathbb{C})$. It will turn out to be useful to also know the Lefschetz decomposition and we describe this as well.

Let X be a smooth projective variety of dimension n, considered as a compact Kähler manifold. Then X may be viewed as a C^∞ manifold of real dimension $2n$. Over each point $x \in X$, the fibre of the real tangent bundle $T_{x,X,\mathbb{R}}$ to X over x is spanned as a real vector space by the derivations $\{\partial/\partial x_j, \partial/\partial y_j\}$. Multiplication by i induces an endomorphism J of $T_{x,X,\mathbb{R}}$ given by $J(\partial/\partial x_j) = \partial/\partial y_j$ and $J(\partial/\partial y_j) = -\partial/\partial x_j$ (we have $J^2 = -\mathrm{id}$)—a so-called almost complex structure on $T_{x,X,\mathbb{R}}$. Extending the scalars to $\mathbb{C}$, it is more convenient to change coordinates and work instead with the complex basis

$$\frac{\partial}{\partial z_j} := \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right), \quad \frac{\partial}{\partial \overline{z_j}} := \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right)$$

of $T_{x,X,\mathbb{C}} := T_{x,X,\mathbb{R}} \otimes \mathbb{C}$. The endomorphism J extends to an endomorphism of $T_{x,X,\mathbb{C}}$; it is clear that the only possible eigenvalues of J are $\pm i$. Let $T^{1,0}_{x,X}$ denote the eigenspace corresponding to the eigenvalue i and $T^{0,1}_{x,X}$ the one corresponding to the eigenvalue $-i$. We have

$$T_{x,X,\mathbb{C}} = T^{1,0}_{x,X} \bigoplus T^{0,1}_{x,X}.$$

One reason for the choice of basis above is that $J(\partial/\partial z_j) = i(\partial/\partial z_j)$ and $J(\partial/\partial \overline{z_j}) = -i(\partial/\partial \overline{z_j})$ so that $\{\partial/\partial z_j\}$ is a basis for $T^{1,0}_{x,X}$ and $\{\partial/\partial \overline{z_j}\}$ a basis for $T^{0,1}_{x,X}$. Dualizing, we have a similar decomposition for the fibre of the complexified cotangent space $\Omega_{x,X,\mathbb{C}} := \Omega_{x,X,\mathbb{R}} \otimes \mathbb{C}$ over x:

$$\Omega_{x,X,\mathbb{C}} = \Omega^{1,0}_{x,X} \bigoplus \Omega^{0,1}_{x,X}.$$

The dual bases for $\Omega^{1,0}_{x,X}$ and $\Omega^{0,1}_{x,X}$ are $\{dz_j := dx_j + i\,dy_j\}$ and $\{d\overline{z_j} := dx_j - i\,dy_j\}$, respectively. Thus we have $\overline{\Omega^{1,0}_{x,X}} = \Omega^{0,1}_{x,X}$, where complex conjugation acts on the complex vector space $\Omega^{1,0}_{x,X}$ in the natural way.

The above constructions extend globally (see [14] pp. 43–55), giving decompositions

$$\bigwedge^r \Omega_{X,\mathbb{C}} = \bigoplus_{p+q=r} \Omega^{p,q}_X, \tag{2.7}$$

for all $r \le n$, where $\Omega_X^{p,q} := \bigwedge^p \Omega_X^{1,0} \otimes \bigwedge^q \Omega_X^{0,1}$. Moreover, we have $\overline{\Omega_X^{p,q}} = \Omega_X^{q,p}$.

Let $A^{p,q}(X)$ denote the C^∞ sections of the bundle $\Omega_X^{p,q}$. In local coordinates, the elements of $A^{p,q}(X)$ look like

$$\sum_{\substack{i_1<i_2<\cdots<i_p \\ j_1<j_2<\cdots<j_q}} f_{i_1,\ldots,i_p,j_1,\ldots,j_q}(z)\ dz_{i_1} \wedge \cdots \wedge dz_{i_p} \wedge d\overline{z_{j_1}} \wedge \cdots \wedge d\overline{z_{j_q}},$$

where $f_{i_1,\ldots,i_p,j_1,\ldots,j_q}$ are C^∞. Define

$$H^{p,q}(X) := \{\alpha \in H^{p+q}(X,\mathbb{C}) : \alpha \text{ may be represented by a form } \omega \in A^{p,q}(X)\}.$$

It is a theorem of Hodge that when X is compact Kähler the decomposition (2.7) on the level of differential forms remains true at the level of cohomology.

Theorem 2.1 (Hodge)**.** *Let (X,ω) be a compact Kähler manifold of dimension n. Then for $0 \le k \le 2n$, $H^k(X,\mathbb{C})$ admits the decomposition*

$$H^k(X,\mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X), \tag{2.8}$$

such that $\overline{H^{p,q}(X)} = H^{q,p}(X)$.

Remark. The decomposition (2.8) is independent of the choice of Kähler metric on X. This is proposition 6.11, pp. 142–143 of [14].

Write $h^{p,q}(X) \coloneqq \dim_{\mathbb{C}} H^{p,q}(X)$. The integers $h^{p,q}(X)$ are often displayed in a diamond

$$\begin{array}{ccccccc} & & & h^{0,0}(X) & & & \\ & & h^{1,0}(X) & & h^{0,1}(X) & & \\ & \iddots & & \vdots & & \ddots & \\ h^{n,0}(X) & & \cdots & & \cdots & & h^{0,n}(X) \\ & \ddots & & \vdots & & \iddots & \\ & & h^{n,n-1}(X) & & h^{n-1,n}(X) & & \\ & & & h^{n,n}(X) & & & \end{array}$$

As $H^{p,q}(X) = \overline{H^{q,p}(X)}$, we have $h^{p,q}(X) = h^{q,p}(X)$, so that the Hodge diamond of X is symmetric about the middle vertical. By Serre duality we have $h^{p,q}(X) = h^{n-p,n-q}(X)$, so that the Hodge diamond of X is symmetric about the middle row.

Example. *Let X be a smooth genus g curve. Then X is connected and has topological Euler characteristic $2-2g$. Using symmetry under complex conjugation, we have $h^{1,0}(X) = h^{0,1}(X)$, so $h^1(X) = h^{1,0}(X)+h^{0,1}(X) = 2h^{1,0}(X)$. Then $2-2g = h^0(X)-h^1(X)+h^2(X) = 2-2h^{1,0}(X)$, so X has the Hodge diamond*

$$\begin{array}{ccc} & 1 & \\ g & & g \\ & 1 & \end{array} .$$

Example. *Let X be a K3 surface. Then X has the Hodge diamond*

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array} .$$

(see [1], Ch. VIII)

A rough idea of the proof of theorem 2.1 is as follows. First, for a compact C^∞ manifold X with Riemannian metric g, there is a bijection between cohomology classes in $H^k(X,\mathbb{R})$ and real-valued differential k-forms $\eta \in A^k(X,\mathbb{R})$ satisfying $\Delta_{d,g}\,\eta = 0$, where $\Delta_{d,g}$ is the Laplacian operator on $A^k(X,\mathbb{R})$ associated to the metric g and the usual operator $d : A^k(X,\mathbb{R}) \longrightarrow A^{k+1}(X,\mathbb{R})$ (the so-called (d,g)-harmonic forms). By extension of scalars, it is immediate that there is a bijection between $H^k(X,\mathbb{C})$ and complex-valued (d,g)-harmonic forms. For a general compact complex manifold with hermitian metric h, the components $\eta^{p,q}$ of a (d,h)-harmonic form η may not themselves be (d,h)-harmonic. However, for compact Kähler manifolds (X,ω) with Kähler metric $h = g - i\omega$, the Laplacians associated to the operators d, ∂ and $\overline{\partial}$ agree up to a constant and this fact is used to show that if $\eta \in A^k(X)$ satisfies $\Delta_{d,h}\,\eta = 0$, then the components $\eta^{p,q}$ of η again satisfy $\Delta_{d,h}\,\eta^{p,q} = 0$. Then the decomposition (2.8) on the level of cohomology follows from the analogous decomposition for harmonic forms. A detailed and complete proof of theorem 2.1 may be found in section 0.6 of [6].

With notation as above, we now turn to the Lefschetz decomposition of the complex cohomology of (X,ω) . For each $0 \le k \le 2n-2$, define the operator

$$L_k : A^k(X) \longrightarrow A^{k+2}(X)$$
$$\alpha \mapsto \alpha \wedge \omega.$$

It follows from the Leibniz rule $d(\alpha \wedge \omega) = (d\alpha) \wedge \omega + (-1)^k \alpha \wedge (d\omega) = (d\alpha) \wedge \omega$ that L_k sends closed forms in $A^k(X)$ to closed forms in $A^{k+2}(X)$ and exact forms to exact forms, hence induces the operator

$$L_k : H^k(X,\mathbb{C}) \longrightarrow H^{k+2}(X,\mathbb{C}).$$

We often omit the subscript k on L_k. We have

Theorem 2.2 (Hard Lefschetz theorem)**.** *Let (X, ω) be a compact Kähler manifold of dimension n. For all $k < n$, the map*

$$L^k : H^{n-k}(X, \mathbb{C}) \longrightarrow H^{n+k}(X, \mathbb{C})$$

is an isomorphism.

Define the $(n-k)$th primitive cohomology group $H^{n-k}_{\text{prim}}(X, \mathbb{C})$ of X is the kernel of the map $L^{k+1} : H^{n-k}(X, \mathbb{C}) \to H^{n+k+2}(X, \mathbb{C})$. Then for any $k < n$, we have the Lefschetz decomposition

$$H^k(X, \mathbb{C}) = \bigoplus_{2r \le k} L^r H^{k-2r}_{\text{prim}}(X, \mathbb{C}).$$

Proof. One proof is a beautiful application of the representation theory of the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$. It may be found in, say, [6] pp. 118–122. $\square$

Finally, let $H^{p,q}_{\text{prim}}(X, \mathbb{C}) \coloneqq H^{p+q}_{\text{prim}}(X, \mathbb{C}) \cap H^{p,q}(X)$ and write $h^{p,q}_{\text{prim}}(X) = \dim_{\mathbb{C}} H^{p,q}_{\text{prim}}(X, \mathbb{C})$. We have the useful

Lemma 2.4. *We have*

$$H^k_{\text{prim}}(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}_{\text{prim}}(X, \mathbb{C})$$

and

$$h^{p,q}_{\text{prim}}(X) = \begin{cases} h^{p,q} - h^{p-1,q-1}, & p+q \le n \\ h^{p,q} - h^{p+1,q+1}, & p+q \ge n \end{cases}.$$

2.3 Intersection theory and integral cohomology $H^\bullet(X, \mathbb{Z})$

The integral cohomology ring $H^\bullet(X, \mathbb{Z})$ of a smooth projective variety X (viewing X as a compact complex manifold) plays a role analogous to that of the Chow ring. By Poincaré duality, the elements of $H^\bullet(X, \mathbb{Z})$ can be interpreted as cycles modulo a certain equivalence relation (namely homological equivalence) and, for any pair of torsion-free cohomology classes $\alpha, \beta \in H^k(X)$, the cup product is Poincaré dual to oriented intersection of cycles (as long as k-cycles X and Y Poincaré dual to α and β intersect transversally—

and this is always possible to arrange when X is C^∞ by picking homologically equivalent classes X' and Y' if necessary).

In this section, we give explicit descriptions of $H^\bullet(\mathbb{P}^n, \mathbb{Z})$, $H^\bullet(\mathbb{P}^a \times \mathbb{P}^b, \mathbb{Z})$ and of the additive structure of $H^\bullet(\mathbb{G}(k,n), \mathbb{Z})$. Explicit knowledge of these rings will be useful in Chapter 3 to determine which of the spaces $\mathbb{P}^n$ and $\mathbb{G}(k,n)$ have a positive-definite intersection form. The description of the intersection form on $H^{(k+1)(n-k)}(\mathbb{G}(n,k), \mathbb{Z})$ will also be used in simple Schubert calculus arguments in Chapter 5. In the last part, we write down the excess intersection formula, which will be used throughout chapter 5.

2.3.1 Integral cohomology of $\mathbb{P}^n$ and $\mathbb{P}^a \times \mathbb{P}^b$

The cell decomposition

$$\mathbb{P}^n \cong (\mathbb{P}^n \smallsetminus \mathbb{P}^{n-1}) \sqcup \cdots \sqcup (\mathbb{P}^1 \smallsetminus \mathbb{P}^0) \sqcup \mathbb{P}^0$$

with $(\mathbb{P}^k \smallsetminus \mathbb{P}^{k-1}) \cong \mathbb{C}^k$ shows that the integral cohomology ring $H^\bullet(\mathbb{P}^n, \mathbb{Z})$ of $\mathbb{P}^n$ is equal to

$$H^\bullet(\mathbb{P}^n, \mathbb{Z}) = \frac{\mathbb{Z}[H]}{(H^{n+1})}.$$

where H is the hyperplane class. The class H has real codimension 2, hence is of degree 2 in the grading and

$$H^k(\mathbb{P}^n, \mathbb{Z}) = \begin{cases} \mathbb{Z}, & 0 \le k \le 2n,\ k \text{ even} \\ 0, & \text{otherwise} \end{cases}.$$

Let Y be a closed smooth subvariety of $\mathbb{P}^n$ of codimension r. Then by above the Poincaré dual class $[Y]$ is equal to $d \cdot H^r$ for some $d \in \mathbb{Z}$. The following lemma gives an interpretation of d in terms of the geometry of Y

Lemma 2.5. *The integer d above is equal to the degree of Y.*

Proof. We interpret the intersection of Y with a generic $\mathbb{P}^{n-r}$ in two different ways. On one hand, this is $e := \deg(Y)$ distinct points. Therefore $[Y \cap \mathbb{P}^{n-r}] = [\text{pt.}] + \cdots + [\text{pt.}] = e\ [\text{pt.}]$, where the middle sum has e summands. On the other hand, by taking generic hyperplane sections,

$$[Y \cap \mathbb{P}^{n-r}] = [Y] \cdot H^{n-r} = d\, H^n = d\,[\text{pt.}]\,,$$

where the last sum has d summands. So $d = \deg(Y)$, as desired. $\square$

Now by Künneth's theorem, we have

$$H^\bullet(\mathbb{P}^a \times \mathbb{P}^b, \mathbb{Z}) = H^\bullet(\mathbb{P}^a, \mathbb{Z}) \otimes H^\bullet(\mathbb{P}^b, \mathbb{Z}) = \frac{\mathbb{Z}[H_1, H_2]}{(H_1^{a+1}, H_2^{b+1})}.$$

Thus

$$H^k(\mathbb{P}^a \times \mathbb{P}^b, \mathbb{Z}) = \begin{cases} \mathbb{Z}\, H_1^k \oplus \mathbb{Z}\, H_1^{k-1} H_2 \oplus \cdots \oplus \mathbb{Z}\, H_1 H_2^{k-1} \oplus \mathbb{Z}\, H_2^k, & 0 \le k \le 2(a+b),\ k \text{ even} \\ 0, & k \text{ odd} \end{cases}$$

One identifies the class H_i $(i = 1, 2)$ as the class of a pullback of a hyperplane under the projection π_i to the i-th factor of $\mathbb{P}^a \times \mathbb{P}^b$.

2.3.2 Additive structure of $H^\bullet(\mathbb{G}(k, n), \mathbb{Z})$

Here we describe the additive structure of the graded pieces $H^i(\mathbb{G}(k, n), \mathbb{Z})$. One reference for this subsection is [6] pp. 193–206.

As an abelian group, each even-codimensional piece of the integral cohomology $H^{2i}(\mathbb{G}(k, n), \mathbb{Z})$ $(0 \le i \le \dim_{\mathbb{C}} \mathbb{G}(k, n))$ of the Grassmannian of k-planes in $\mathbb{P}^n$ is freely generated by cycles $\sigma_{a_0, \ldots, a_k}$ where $0 \le a_0 \le \cdots \le a_k \le (n-k)$ are integers such that $\sum a_j = i$. The indexing sequences $a_0, \ldots, a_k$ may be thought of as Young diagrams fitting inside a $(k+1) \times (n-k)$ rectangle. We now describe the cycles $\sigma_{a_0, \ldots, a_r}$.

For a moment, we work with the Grassmannian $G(k+1, n+1)$ of $(k+1)$-dimensional subspaces of $\mathbb{C}^{n+1}$. Let $B = \{e_1, \ldots, e_{n+1}\}$ be a basis of $\mathbb{C}^{n+1}$. A $(k+1)$-dimensional subspace of $\mathbb{C}^{n+1}$ is determined by a choice of $k+1$ linearly independent vectors in $\mathbb{C}^{n+1}$.

Such choices are in bijection with full-rank $(k+1)\times(n+1)$ matrices

$$\begin{pmatrix} v_{1,1} & \cdots & v_{1,n+1} \\ \vdots & \ddots & \vdots \\ v_{r+1,1} & \cdots & v_{r+1,n+1} \end{pmatrix}.$$

Let $\mathrm{GL}(n,\mathbb{C})$ be the group of invertible $n\times n$ matrices over $\mathbb{C}$. Two choices of $k+1$ linearly independent vectors corresponding to matrices M and M' as above determine the same subspace of $\mathbb{C}^{n+1}$ if and only if there exists $g \in \mathrm{GL}(k+1,\mathbb{C})$ such that $g \cdot M = M'$, with g acting on M by matrix multiplication. That is, M and M' determine the same subspace if and only if M and M' are in the same orbit of the natural action of $\mathrm{GL}(k+1,\mathbb{C})$ on the set of full-rank $k+1\times n+1$ matrices. Linear algebra gives a unique representative of each orbit under the above action: matrices in reduced row-echelon form. This gives a well-defined map from $k+1$-dimensional linear subspaces of $\mathbb{C}^{n+1}$ to full-rank $(k+1)\times(n+1)$ matrices. The generic such matrix will look like

$$\begin{pmatrix} 1 & \cdots & 0 & * & \cdots & * \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 1 & * & \cdots & * \end{pmatrix},$$

where $*$ denote arbitrary complex numbers. Let a_r be the number of cells that the pivot on the $(r-1)$-st row moves from its generic position. Then clearly $0 \le a_0 \le \cdots \le a_k \le (n+1)-(k+1) = n-k$. We define

$$S^B_{a_0,\dots,a_k} := \left\{ \Lambda \in \mathbb{G}(k,n) : \begin{array}{c} \text{RREF of the matrix representing the } k+1 \text{ dimensional subspace} \\ \text{associated to } \Lambda \text{ has sequence type } a_0,\dots,a_k \end{array} \right\}$$

(we have switched back to working with $\mathbb{G}(k,n)$) and take

$$\sigma_{a_0,\dots,a_k} := \left[\ \overline{S^B_{a_0,\dots,a_k}}\ \right]$$

to be the cycle class in $H^{2\sum a_i}(\mathbb{G}(k,n),\mathbb{Z})$ of the Zariski-closure of $S^B_{a_0,\ldots,a_r}$ in $\mathbb{G}(k,n)$. It is implicit in our notation that the cycle class $\sigma_{a_0,\ldots,a_k}$ is independent of the initial choice of basis B. This will be clear from a more geometric description given below of the subvarieties $S_{a_0,\ldots,a_k}$ as k-planes in $\mathbb{P}^n$ intersecting a fixed complete flag in dimensions specified by the sequence $a_0,\ldots,a_k$.

We remark that with the above description, it is clear that each $S_{a_0,\ldots,a_k}$ is affine; in fact, $S_{a_0,\ldots,a_k} \cong \mathbb{A}^{(k+1)(n-k)-\sum a_i}$. Moreover, because the action of $\mathrm{GL}(k+1,\mathbb{C}^{n+1})$ induces a decomposition of the set of full-rank $(k+1)\times(n+1)$ matrices into a disjoint union of orbits, we have described a decomposition of the Grassmannian $\mathbb{G}(k,n)$ into a disjoint union of (locally closed) affine subvarieties $S_{a_0,\ldots,a_k}$. Hence $S_{a_0,\ldots,a_k}$ give a cellular decomposition of $\mathbb{G}(k,n)$. Because each $S_{a_0,\ldots,a_k}$ has even real dimension, each boundary map in the CW complex is the zero map. It follows that the cycles $\sigma_{a_0,\ldots,a_k}$ indeed generate the cohomology in their respective codimension.

The choice of basis $\{e_1,\ldots,e_{n+1}\}$ made above leads to a more geometric description of the cycles $\sigma_{a_0,\ldots,a_k}$. We have a natural complete flag

$$\langle e_{n+1}\rangle \subset \langle e_n, e_{n+1}\rangle \subset \cdots \subset \langle e_1,\ldots,e_{n+1}\rangle$$

or, projectivizing, a complete projective flag

$$\mathbb{P}^0 \subset \mathbb{P}^1 \subset \cdots \subset \mathbb{P}^n$$

where $\mathbb{P}^m \cong \mathbb{P}(\langle e_{n+1-m},\ldots,e_{n+1}\rangle)$. We give a few examples of how the subvarieties $S^B_{a_0,\ldots,a_k}$ of $\mathbb{G}(k,n)$ may be defined by conditions on the dimensions of the intersection of a k-plane $\Lambda \in S^B_{a_0,\ldots,a_k}$ with the above flags.

Example. *We work with the Grassmannian $\mathbb{G}(1,3) = G(2,4)$ of lines in $\mathbb{P}^3$. Fix a basis $B = \{e_1,\ldots,e_4\}$ of $\mathbb{C}^4$. We expect a generic 2-dimensional subspace of $\mathbb{C}^4$ to intersect the elements $\langle e_4\rangle$, $\langle e_3,e_4\rangle$, $\langle e_2,e_3,e_4\rangle$, $\langle e_1,e_2,e_3,e_4\rangle$ of the flag associated to B in 0, 0, 1 and*

2 dimensional subspaces, respectively. We study the two-dimensional subspaces contained in the strata $S^B_{a_0,a_1}$ *for various sequences* a_0, a_1*:*

(0,0): The RREF is

$$\begin{pmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \end{pmatrix}.$$

Since vectors in $\langle e_3, e_4\rangle$ *are of the form* $(0,0,*,*)$*, we have* $\langle e_3, e_4\rangle \cap \Lambda = \varnothing$*, for any* $\Lambda \in S^B_{0,0}$*. Moreover,* $\langle e_2, e_3, e_4\rangle \cap \Lambda = \langle (0,1,*,*)\rangle$ *and* $\langle e_3, e_4\rangle \cap \Lambda = \Lambda$*. As expected, the sequence* $0,0$ *corresponds to generic 2-planes.*

(0,1): The RREF is

$$\begin{pmatrix} 1 & * & 0 & * \\ 0 & 0 & 1 & * \end{pmatrix}.$$

The dimensions of intersections of a 2-dimensional subspace $\Lambda \in S^B_{0,1}$ *with the flag elements are*

$\langle e_4\rangle$	$\langle e_3, e_4\rangle$	$\langle e_2, e_3, e_4\rangle$	$\langle e_1, e_2, e_3, e_4\rangle$
$\varnothing$	$\langle (0,0,1,*)\rangle$	$\langle (0,0,1,*)\rangle$	Λ
$\dim 0$	$\dim 1$	$\dim 1$	$\dim 2$

In the projective picture, every line $\Lambda \in S^B_{0,1}$ *is constrained to intersect the line* $\ell = \mathbb{P}(\langle e_3, e_4\rangle)$ *in a point.*

(0,2): The RREF is

$$\begin{pmatrix} 1 & * & * & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The dimensions of intersections of a 2-dimensional subspace $\Lambda \in S^B_{0,2}$ *with the flag* *elements are*

$\langle e_4\rangle$	$\langle e_3, e_4\rangle$	$\langle e_2, e_3, e_4\rangle$	$\langle e_1, e_2, e_3, e_4\rangle$
$\langle(0,0,0,1)\rangle$	$\langle(0,0,0,1)\rangle$	$\langle(0,0,0,1)\rangle$	Λ
$\dim 1$	$\dim 1$	$\dim 1$	$\dim 2$

Projectively, every line $\Lambda \in S^B_{0,2}$ is constrained to contain the point $\mathbb{P}(\langle e_4\rangle)$.

(1,1): The RREF is

$$\begin{pmatrix} 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \end{pmatrix}.$$

The dimensions of intersections of a 2-dimensional subspace $\Lambda \in S^B_{1,1}$ with the flag elements are

$\langle e_4\rangle$	$\langle e_3, e_4\rangle$	$\langle e_2, e_3, e_4\rangle$	$\langle e_1, e_2, e_3, e_4\rangle$
$\varnothing$	$\langle(0,0,1,*)\rangle$	Λ	Λ
$\dim 1$	$\dim 1$	$\dim 2$	$\dim 2$

Projectively, every line $\Lambda \in S^B_{1,1}$ is constrained to lie in the 2-plane $\mathbb{P}(\langle e_2, e_3, e_4\rangle)$

The cases $(1,2)$ and $(2,2)$ are similar.

In general, with respect to a choice of complete projective flag $\mathcal{F} = \mathbb{P}^0 \subset \cdots \subset \mathbb{P}^n$, define

$$S^{\mathcal{F}}_{a_0,\dots,a_k} := \left\{\Lambda \,\middle|\, \dim(\Lambda \cap \mathbb{P}^{n-k+m-a_m}) = m,\ m = 0,1,\dots,k\right\}.$$

Then

$$\overline{S^{\mathcal{F}}_{a_0,\dots,a_k}} = \left\{\Lambda \,\middle|\, \dim(\Lambda \cap \mathbb{P}^{n-k+m-a_m}) \geq m,\ m = 0,1,\dots,k\right\}.$$

As suggested by the examples above, if $B = \{e_1, \dots, e_{n+1}\}$ is a choice of basis for $\mathbb{C}^{n+1}$ and $\mathcal{F}$ is the flag $\mathbb{P}(\langle e_{n+1}\rangle) \subset \cdots \subset \mathbb{P}(\langle e_1, \dots, e_{n+1}\rangle)$ determined by this basis as above, then $S^B_{a_0,\dots,a_k} = S^{\mathcal{F}}_{a_0,\dots,a_k}$. A proof may be found in [6], pp. 193–197.

Let $\mathrm{PGL}(n+1, \mathbb{C})$ be the group of invertible $(n+1)\times(n+1)$ matrices over $\mathbb{C}$, quotiented by the subgroup of scalar transformations (diagonal matrices with equal nonzero entries).

It is well-known that $\mathrm{PGL}(n+1,\mathbb{C}) = \mathrm{Aut}(\mathbb{P}^n)$, the group of automorphisms of $\mathbb{P}^n$, with matrices in $\mathrm{PGL}(n+1,\mathbb{C})$ acting by multiplication on the left on the column vectors of homogeneous coordinates of $\mathbb{P}^n$. The following lemma is useful for showing that certain cohomology classes are equal:

Lemma 2.6. $\mathrm{PGL}(n+1,\mathbb{C})$ *is path-connected.*

Proof. Let M and M' be two matrices in $\mathrm{PGL}(n+1,\mathbb{C})$. Take the pencil of matrices

$$\lambda M + \mu M', \quad [\lambda : \mu] \in \mathbb{P}^1.$$

Now let $F(\lambda,\mu) := \det(\lambda M + \mu M')$. Then F is a degree $n+1$ homogeneous form in λ and μ, hence we can consider its vanishing set in $\mathbb{P}^1$. As $F(1,0) \neq 0$, F does not vanish identically, so that the vanishing set $V(F)$ consists of $(n+1)$ points (counted with multiplicity), say $p_1,\dots,p_{n+1}$, with none of the p_i equal to $[1:0]$ or $[0:1]$. But $Y := \mathbb{P}^1 \setminus \{p_1,\dots,p_{n+1}\}$ is path-connected and a path from $[1:0]$ to $[0:1]$ in Y induces a path from M to M' in $\mathrm{PGL}(n+1,\mathbb{C})$. □

Now given any two complete projective flags in $\mathbb{P}^n$, there exists a $g \in \mathrm{PGL}(n+1,\mathbb{C})$ taking one to the other (this may be seen from the well-known fact that given any two sets of $n+2$ points in linearly independent position in $\mathbb{P}^n$, there exists a $g \in \mathrm{PGL}(n+1,\mathbb{C})$ taking the first set to the second—the statement for $n+1$ points suffices for the case of the flag). Moreover, as $\mathrm{PGL}(n+1,\mathbb{C})$ is path-connected, there exists a continuous map $[0,1] \to \mathrm{PGL}(n+1,\mathbb{C})$, say $t \mapsto g(t)$, where $g(0) = \mathrm{id}$ and $g(1) = g$. The action of $g(t)$ on the first flag induces a continuous deformation of the first flag to the second. Since integral cohomology is a discrete invariant, it follows that the cycle class $\sigma_{a_0,\dots,a_k}$ is independent of choice of flag. It follows in particular that $\sigma_{a_0,\dots,a_k}$ is independent of the choice of basis B made to define $S^B_{a_0,\dots,a_k}$.

The above completes our description of the even-codimensional cohomology groups. Because the Grassmannian $\mathbb{G}(k,n)$ is the disjoint union of the $S_{a_0,\dots,a_k}$, each of which has even real dimension, in particular $H^{2i+1}(\mathbb{G}(k,n),\mathbb{Z}) = 0$ for each $0 \le i < \dim_{\mathbb{C}} \mathbb{G}(k,n)$. The description of the multiplication on $H^\bullet(\mathbb{G}(k,n),\mathbb{Z})$ in terms of the generators $\sigma_{a_0,\dots,a_k}$ is intricate, but well-understood. This is the so-called Schubert calculus—a description may be found in [6], pp. 197–206. For our purposes, it is enough to describe the multiplication rule for cycles in complementary codimension:

Lemma 2.7. *Let $\sigma_{a_0,\dots,a_k}$ and $\sigma_{b_0,\dots,b_k}$ be cycles with $\sum a_i + \sum b_i = \dim_{\mathbb{C}} \mathbb{G}(k,n) = (k+1)(n-k)$. Then*

$$\sigma_{a_0,\dots,a_k} \cdot \sigma_{b_0,\dots,b_k} = \begin{cases} 1, & \text{if } a_0 = n-k-b_k, \dots, a_k = n-k-b_0 \\ 0, & \text{otherwise} \end{cases} \tag{2.9}$$

Proof. See [6] pp. 197–198. □

In the language of Young diagrams, two generating cycles $\sigma_{a_0,\dots,a_k}$ and $\sigma_{b_0,\dots,b_k}$ of complementary dimension have product 1 in $H^\bullet(\mathbb{G}(k,n),\mathbb{Z})$ if the Young diagrams associated to the sequences $a_0,\dots,a_k$ and $b_0,\dots,b_k$ are complementary (with respect to a $(k+1)\times(n-k)$ rectangle) and product 0 otherwise.

Example. *Consider the Grassmannian $\mathbb{G}(1,3)$ of lines in $\mathbb{P}^3$. There are a total of six sequences a_0, a_1 with $0 \le a_0 \le a_1 \le 2$:*

Codimension in $\mathbb{G}(1,3)$	a_0, a_1
0	0, 0
1	0, 1
2	0, 2
2	1, 1
3	1, 2
4	2, 2

The cycles of complementary dimension are $\sigma_{0,0}$ and $\sigma_{2,2}$; $\sigma_{0,1}$ and $\sigma_{1,2}$; $\sigma_{0,2}$ and itself; $\sigma_{1,1}$ and itself; and $\sigma_{0,2}$ and $\sigma_{1,1}$.

Let $p \subset \ell \subset h \subset s$ be a fixed complete projective flag, with $p \cong \mathbb{P}^0$ a point, $\ell \cong \mathbb{P}^1$ a line, $h \cong \mathbb{P}^2$ a hyperplane and $s \cong \mathbb{P}^3$.

- *The cycle $\sigma_{0,0}$ corresponds to lines in $\mathbb{P}^3$ intersecting the flag generically. In particular, each such line will not intersect ℓ. But the cycle $\sigma_{2,2}$ corresponds to point of $\mathbb{G}(1,3)$ corresponding to ℓ. Hence $\sigma_{0,0} \cdot \sigma_{2,2} = 0$.*

- *Since the cycle classes σ_{a_0,a_1} are independent of the choice of flag, we can consider $\sigma_{1,2}$ with respect to any complete projective flag $p' \subset \ell' \subset h' \subset s'$. Choose h' intersecting the line ℓ of the original flag in a point (a generic choice of hyperplane will do) and choose p' to be a point on h' different from the point of intersection of ℓ and h'. Now the cycle $\sigma_{0,1}$ corresponds to lines in $\mathbb{P}^3$ intersecting ℓ in a point. The cycle $\sigma_{1,2}$ corresponds to lines through p' contained in h'. There is a unique line satisfying both conditions (the line through p' and the point of intersection of ℓ with h'). Hence $\sigma_{0,1} \cdot \sigma_{1,2} = 1$.*

- *The cycle $\sigma_{0,2}$ corresponds to lines in $\mathbb{P}^3$ containing p. Making a different choice p and p' of the points in the two flags determining $\sigma_{0,2}$ shows that $\sigma_{0,2}^2$ consists of the unique line containing p and p'. Hence $\sigma_{0,2}^2 = 1$.*

- *The cycle $\sigma_{1,1}$ corresponds to lines in $\mathbb{P}^3$ contained in h. Making different choices h and h' of the hyperplanes in the two flags determining $\sigma_{1,1}$ shows that $\sigma_{1,1}^2$ consists of the unique line contained in the intersection of h and h'. Hence $\sigma_{1,1}^2 = 1$.*

- *Picking p in the flag determining $\sigma_{0,2}$ to be a point off the hyperplane h' in the flag determining $\sigma_{1,1}$ shows that $\sigma_{0,2} \cdot \sigma_{1,1} = 0$.*

2.3.3 Excess intersection formula

For a complex rank r vector bundle E on a variety X, let $c_t(E) = \sum_{k=0}^{r} c_k(E)t^k$ denote the Chern polynomial of E and $c(E) \coloneqq c_t(E)|_{t=1} = \sum_{k=0}^{r} c_k(E)$ denote the total Chern class. The following theorem will be useful in chapter 5 for computing cup products of configurations of projective varieties not intersecting in the expected dimension:

Theorem 2.3 (Excess Intersection Formula)**.** *Let $X \subset \mathbb{P}^m$ be a smooth variety of dimension $n = k\ell$. Let $Y_1, \dots, Y_\ell$ be smooth closed subvarieties of X, each of dimension k, and suppose that $Z = Y_1 \cap \dots \cap Y_\ell$ is again smooth. Then*

$$[Y_1] \cdot [Y_2] \cdot \dots \cdot [Y_\ell] = \int_Z \frac{c(N_{Y_1/X}\big|_Z) \cdots c(N_{Y_\ell/X}\big|_Z)}{c(N_{Z/X})}.$$

Proof. See prop. 9.1.1, pg. 154 of [5]. □

Applying theorem 2.3 with $n = m$, $X = \mathbb{P}^n$ and $Y_1 = Y_2 = \dots = Y_\ell$ immediately gives

Corollary 2.2. *Let $Y \subset \mathbb{P}^n$ be a smooth variety of dimension $k = n/\ell$. Then*

$$[Y]^\ell = \int_Y c(N_{Y/\mathbb{P}^n})^{\ell-1}.$$

Before giving examples, we write down the following lemma that in particular will be useful in computing normal bundles to smooth complete intersections:

Lemma 2.8. *Let X be a variety and let E be a complex rank r vector bundle on X. If $s \in H^0(X, E)$ is a global section of E such that $Y = \mathrm{Zeros}(s) := \{x \in X : s_x = 0 \in E_x\}$ has codimension r in X, then*

$$N_{Y/X} = E|_Y .$$

Proof. Let $\mathcal{I}_Y$ be the ideal sheaf of Y. We identify the normal bundle $N_{Y/X}$ with the dual sheaf to the quotient sheaf $\mathcal{I}_Y/\mathcal{I}_Y^2$.

Let $\{U_\alpha\}_{\alpha\in I}$ be a local trivialization of $E \to X$ and let $M_{\alpha\beta}$ be the associated transition matrices. Write $s_\alpha = s|_{U_\alpha} = (f_1^\alpha, \dots, f_r^\alpha)$ for any $\alpha \in I$. By definition of the transition matrices, we have

$$\begin{pmatrix} f_1^\beta \\ \vdots \\ f_r^\beta \end{pmatrix} = M_{\alpha\beta} \begin{pmatrix} f_1^\alpha \\ \vdots \\ f_r^\alpha \end{pmatrix}$$

for all $\alpha, \beta \in I$.

By hypothesis, the restriction $\mathcal{I}_Y|_{U_\alpha}$ is generated by $f_1^\alpha, \ldots, f_r^\alpha$ as an $\mathcal{O}_X|_{U_\alpha}$-module. We have that $\mathcal{I}_Y/\mathcal{I}_Y^2$ is a free $\mathcal{O}_Y$-module of rank r and $f_1^\alpha, \ldots, f_r^\alpha$ are a $\mathcal{O}_Y$-basis for $\mathcal{I}_Y/\mathcal{I}_Y^2$ over U_α.

For any $\alpha, \beta \in I$, let $t_\alpha = g_1^\alpha f_1^\alpha + \cdots + g_r^\alpha f_r^\alpha$, $g_i^\alpha \in \mathcal{O}_Y(U_\alpha)$ and $t_\beta = g_1^\beta f_1^\beta + \cdots + g_r^\beta f_r^\beta$, $g_i^\beta \in \mathcal{O}_Y(U_\beta)$ be sections of $\mathcal{I}_Y/\mathcal{I}_Y^2$ over U_α and U_β, respectively. We may write

$$t_\alpha = \begin{pmatrix} g_1^\alpha \cdots g_r^\alpha \end{pmatrix} \begin{pmatrix} f_1^\alpha \\ \vdots \\ f_r^\alpha \end{pmatrix} \quad \text{and} \quad t_\beta = \begin{pmatrix} g_1^\beta \cdots g_r^\beta \end{pmatrix} \begin{pmatrix} f_1^\beta \\ \vdots \\ f_r^\beta \end{pmatrix} = \begin{pmatrix} g_1^\alpha \cdots g_r^\alpha \end{pmatrix} M_{\alpha\beta} \begin{pmatrix} f_1^\alpha \\ \vdots \\ f_r^\alpha \end{pmatrix},$$

so that $\begin{pmatrix} g_1^\alpha \cdots g_r^\alpha \end{pmatrix} = \begin{pmatrix} g_1^\beta \cdots g_r^\beta \end{pmatrix} M_{\alpha\beta}$ or, taking transpose,

$$\begin{pmatrix} g_1^\alpha \\ \vdots \\ g_r^\alpha \end{pmatrix} = M_{\alpha\beta}^t \begin{pmatrix} g_1^\beta \\ \vdots \\ g_r^\beta \end{pmatrix}.$$

This shows that, if $N_{\alpha\beta}$ are the transition matrices for $\mathcal{I}_Y/\mathcal{I}_Y^2$ over Y for the local trivialization $\{U_\alpha \cap Y\}_{\alpha \in I}$, we have $N_{\alpha\beta} = M_{\alpha\beta}^t\big|_Y$. Then $\mathcal{I}_Y/\mathcal{I}_Y^2 \cong E^*|_Y$. The statement of the lemma follows by taking duals. $\square$

Corollary 2.3. *Let $X = X(\mathbf{d})$ be a smooth complete intersection of type $\mathbf{d} = (d_1, \ldots, d_r)$ in $\mathbb{P}^N$. Then we have*

$$N_{X/\mathbb{P}^N} = \Big(\mathcal{O}_{\mathbb{P}^N}(d_1) \oplus \mathcal{O}_{\mathbb{P}^N}(d_2) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^N}(d_r)\Big)\Big|_X$$

Proof. The global sections of the bundle $\mathcal{O}_{\mathbb{P}^N}(d)$ are exactly the homogeneous degree d forms on $\mathbb{P}^N$. Moreover, if E and F are vector bundles on X and $s \in H^0(X,E), t \in H^0(X,F)$ are global sections, then $s \oplus t \in H^0(X,E) \oplus H^0(X,F) \cong H^0(X, E \oplus F)$ and $\text{Zeros}(s \oplus t) = \text{Zeros}(s) \cap \text{Zeros}(t)$. Because X is a complete intersection by hypothesis, we have $X = \cap_{i=1}^r \text{Zeros}(F_i) = \text{Zeros}(F_1 \oplus \cdots \oplus F_r)$, for some $F_i \in H^0(X, \mathcal{O}_{\mathbb{P}^N}(d_i))$. Now apply lemma 2.8. □

Examples.

- *Look at three 2-planes Λ_1, Λ_2, Λ_3 in $\mathbb{P}^3$ intersecting in a line ℓ. By corollary 2.3, $N_{\Lambda_i/\mathbb{P}^3} = \mathcal{O}_{\mathbb{P}^3}(1)\big|_{\Lambda_i}$ and $N_{\ell/\mathbb{P}^3} = \mathcal{O}_{\mathbb{P}^3}(1) \oplus \mathcal{O}_{\mathbb{P}^3}(1)\big|_{\ell}$, so that $c(N_{\Lambda_i/\mathbb{P}^3}) = (1 + H|_{\Lambda_i})$ and $c(N_{\ell/\mathbb{P}^3}) = (1 + H|_{\ell})^2 = (1+p)^2$, where p is the class of a point in $H^\bullet(\ell, \mathbb{Z}) \cong H^\bullet(\mathbb{P}^1, \mathbb{Z})$. Then excess intersection formula gives*

$$[\Lambda_1]\cdot[\Lambda_2]\cdot[\Lambda_3] = \int_\ell \frac{c(N_{\Lambda_1/\mathbb{P}^3}\big|_\ell)\cdot c(N_{\Lambda_2/\mathbb{P}^3}\big|_\ell)\cdot c(N_{\Lambda_3/\mathbb{P}^3}\big|_\ell)}{c(N_{\ell/\mathbb{P}^3})} = \int_\ell \frac{(1+p)^3}{(1+p)^2} = \int_\ell 1 + p = 1.$$

- *Look at three quadrics Q_1, Q_2, Q_3 in $\mathbb{P}^3$ with $Q_1 \cap Q_2 \cap Q_3 = C$, a twisted cubic curve. (For example, we may take $Q_1 = Z_0Z_2 - Z_1^2$, $Q_2 = Z_0Z_3 - Z_1Z_2$, $Q_3 = Z_1Z_3 - Z_2^2$). We have $C \cong \mathbb{P}^1$ (being the image of the Veronese embedding of $\mathbb{P}^1$ in $\mathbb{P}^3$), so that $H^\bullet(C,\mathbb{Z}) \cong H^\bullet(\mathbb{P}^1, \mathbb{Z})$. On the other hand, C is embedded as a curve of degree 3, so in particular for any line bundle $\mathcal{O}_{\mathbb{P}^3}(k)$ on $\mathbb{P}^3$, we have $\mathcal{O}_{\mathbb{P}^3}(k)\big|_C = \mathcal{O}_{\mathbb{P}^1}(3k)$. By corollary 2.3, $N_{Q_i/\mathbb{P}^3} = \mathcal{O}_{\mathbb{P}^3}(2)\big|_{Q_i}$ for $i = 1,2,3$, so that $N_{Q_i/\mathbb{P}^3}\big|_C = \mathcal{O}_{\mathbb{P}^1}(6)$ and*

$$c(N_{Q_i/\mathbb{P}^3}\big|_C) = (1+6p),$$

where p is the class of a point in $H^\bullet(\mathbb{P}^1, \mathbb{Z})$.

We now determine $c(N_{C/\mathbb{P}^3})$. First, for any $n \geq 1$ we have the Euler exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n} \longrightarrow \bigoplus_{i=1}^{n+1} \mathcal{O}_{\mathbb{P}^n}(1) \longrightarrow T_{\mathbb{P}^n} \longrightarrow 0.$$

Then by multiplicativity of Chern polynomials in short exact sequences, we have

$$c_t(\bigoplus_{i=1}^{n+1} \mathcal{O}_{\mathbb{P}^n}(1)) = c_t(\mathcal{O}_{\mathbb{P}^n}) \cdot c_t(T_{\mathbb{P}^n}) = c_t(T_{\mathbb{P}^n}),$$

as $c_t(\mathcal{O}_{\mathbb{P}^n}) = 1$*, identifying* $\mathcal{O}_{\mathbb{P}^n}$ *with the sheaf of sections of the trivial line bundle over* $\mathbb{P}^n$*. Then* $c_t(\oplus_{i=1}^{n+1}\mathcal{O}_{\mathbb{P}^n}(1)) = c_t(\mathcal{O}_{\mathbb{P}^n}(1))^{n+1} = (1 + (n+1)H\,t + O(t^2))$*, where* $H \coloneqq c_1(\mathcal{O}_{\mathbb{P}^n}(1))$*. Comparing coefficients of* t*, we find* $c_1(T_{\mathbb{P}^n}) = (n+1)H$*.*

On C*, we have the normal bundle exact sequence (using again that* $C \cong \mathbb{P}^1$*)*

$$0 \longrightarrow T_{\mathbb{P}^1} \longrightarrow T_{\mathbb{P}^3}\big|_C \longrightarrow N_{C/\mathbb{P}^3} \longrightarrow 0,$$

so that

$$c(N_{C/\mathbb{P}^3}) = \frac{c(T_{\mathbb{P}^3}\big|_C)}{c(T_{\mathbb{P}^1})} = \frac{(1 + 4\,H|_C)}{(1+2p)} = \frac{(1+12p)}{(1+2p)} = 1 + 10p.$$

By excess intersection formula,

$$\begin{aligned}
[Q_1]\cdot[Q_2]\cdot[Q_3] &= \int_C \frac{c(N_{Q_1/\mathbb{P}^3}\big|_C)\cdot c(N_{Q_2/\mathbb{P}^3}\big|_C)\cdot c(N_{Q_3/\mathbb{P}^3}\big|_C)}{c(N_{C/\mathbb{P}^3})} \\
&= \int_C \frac{(1+6p)^3}{(1+10p)} \\
&= \int_C (1 + 3\cdot 6p - 10p) \\
&= 8.
\end{aligned}$$

We conclude this subsection with the following calculation, which is useful for excess intersection computations in the case the ambient variety X is a proper subvariety of $\mathbb{P}^n$:

Lemma 2.9. *Let* Z *be a smooth projective variety and* $W \subset Y$ *closed smooth subvarieties of* Z*. Then we have*

$$c_t(N_{W/Z}) = c_t(N_{W/Y})\cdot c_t(N_{Y/Z}\big|_W) \quad or \quad c_t(N_{W/Y}) = \frac{c_t(N_{W/Z})}{c_t(N_{Y/Z}\big|_W)}.$$

Proof. We have the normal bundle exact sequence

$$0 \longrightarrow T_Y \longrightarrow T_Z|_Y \longrightarrow N_{Y/Z} \longrightarrow 0$$

of vector bundles on Y and inclusions

$$T_W \longrightarrow T_Y|_W \quad \text{and} \quad T_W \longrightarrow T_Z|_W$$

of vector bundles on W.

Restricting the $Y \subset Z$ normal bundle exact sequence to W and combining with the above inclusions, we obtain the commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & T_W \arrow[r, "\mathrm{id}"] \arrow[d] & T_W \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & T_Y|_W \arrow[r] \arrow[d] & T_Z|_W \arrow[r] \arrow[d] & N_{Y/Z}\big|_W \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\
0 \arrow[r] & N_{W/Y} \arrow[r] \arrow[d] & N_{W/Z} \arrow[r] \arrow[d] & N_{Y/Z}\big|_W \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

where the top two rows and all of the columns are exact. It then follows by the nine lemma that the bottom row is exact. Because total Chern classes are multiplicative over short exact sequences, this proves the lemma. □

Chapter 3

Examples of smooth varieties with positive-definite intersection form

The descriptions of the integral cohomology ring $H^\bullet(\mathbb{P}^n, \mathbb{Z})$ and the additive structure of the integral cohomology ring $H^\bullet(\mathbb{G}(k,n), \mathbb{Z})$ in section 2.3 allow us to completely determine which of the projective spaces $\mathbb{P}^n$ and Grassmannians $\mathbb{G}(k,n)$ have a positive-definite intersection form on their middle cohomology. We also include a sketch of classification of smooth projective surfaces with positive-definite cup-product form, which will come into one step of the argument in chapter 4.

3.1 Projective spaces

Consider the intersection form on $H^n(\mathbb{P}^n)$. We have

$$H^\bullet(\mathbb{P}^n, \mathbb{Z}) = \frac{\mathbb{Z}[H]}{(H^{n+1})},$$

where H is the hyperplane class, hence of weight 2 in the grading. When n is odd, $H^n(X, \mathbb{Z}) = 0$, so the only interesting case is n even. In the latter case, $H^n(X, \mathbb{Z})$ is a free rank one $\mathbb{Z}$-module with generator $H^{n/2}$. Moreover, we have

$$H^{n/2} \cdot H^{n/2} = \deg(H_1 \cap H_2 \cap \cdots \cap H_n) = 1,$$

where H_i, $1 \leq i \leq n$, are general hyperplanes in $\mathbb{P}^n$. This shows

Proposition 3.1. *The projective space $\mathbb{P}^n$ has a positive-definite intersection form for all n.*

3.2 Grassmannians

Because we have identifications $\mathbb{G}(0,n) = \mathbb{P}^n$ and $\mathbb{G}(n-1,n) = \mathbb{P}^{n*}$, we focus on the Grassmannians $\mathbb{G}(k,n)$ with $0 < k < (n-1)$. We have $\dim \mathbb{G}(k,n) = (k+1)(n-k)$. Once again, the middle cohomology $H^{(k+1)(n-k)}(\mathbb{G}(k,n),\mathbb{Z})$ is only nonzero when $(k+1)(n-k)$ is even.

Proposition 3.2. *The following is a complete list of even-dimensional Grassmannians $\mathbb{G}(k,n)$, $0 < k < (n-1)$, with positive-definite intersection form:*

i). $\mathbb{G}(1,n)$, $n > 1$ arbitrary and

ii). $\mathbb{G}(n-2,n)$, $n > 1$ arbitrary

Proof. By lemma 2.7, each generating class $\sigma_{a_0,\ldots,a_k} \in H^{(k+1)(n-k)}(\mathbb{G}(k,n),\mathbb{Z})$ has exactly one dual generating class, where $\sigma_{a_0,\ldots,a_k}$ is dual to $\sigma_{a'_0,\ldots,a'_k}$ if $\sigma_{a_0,\ldots,a_k} \cdot \sigma_{a'_0,\ldots,a'_k} = 1$ and $\sigma_{a_0,\ldots,a_k} \cdot \sigma_{a''_0,\ldots,a''_k} = 0$ for any $(a''_0,\ldots,a''_k) \neq (a'_0,\ldots,a'_k)$. The key to the proof are the following two observations.

First, if all of the generating cycles of the midddle cohomology are self-dual, then the intersection lattice on the middle cohomology is positive-definite (indeed, equal to the identity matrix of size equal to the rank of $H^{(k+1)(n-k)}(\mathbb{G}(k,n),\mathbb{Z})$).

Second, if there exists a single generator $\sigma_{a_0,\ldots,a_k} \in H(\mathbb{G}(k,n),\mathbb{Z})$ that is not self-dual, then the intersection form is not positive-definite on $H^{(k+1)(n-k)}(\mathbb{G}(k,n),\mathbb{Z})$. Indeed, denoting this class by α and the dual class $\sigma_{n-k-a_k,\ldots,n-k-a_0}$ by β, we have

$$(\alpha - \beta) \cdot (\alpha - \beta) = 0 - 1 - 1 + 0 = -2.$$

We show that for all $\mathbb{G}(k,n)$ not of the types i) and ii), there exists a non-self-dual cycle. There are many ways to see this. Since $(k+1)(n-k)$ must be even, at least one of $(k+1)$ and $(n-k)$ is even, say $(n-k)$. Then $(n-k) \geq 4$, and the sequence $(\frac{n-k}{2},\ldots,\frac{n-k}{2})$ satisfies $a_0 \leq \cdots \leq a_{k+1}$ and $\sum a_i = (k+1)(n-k)/2$. The bottom row contains $a_0 \geq 2$ boxes. Let $(a_0',\ldots,a_k') = (a_0 - 2, a_1, \cdots, a_{k-1} + 1, a_k + 1)$. Then $n - k - a_k' \neq a_0'$. If $(n-k)$ is not even, repeat the previous construction in a $(n-k) \times (k+1)$ box and take the conjugate diagram fitting inside a $(k+1) \times (n-k)$ box.

Finally, we check that indeed for cases i) and ii) all of the generating cycles of the middle cohomology are self-dual.

i). Consider $\mathbb{G}(1,n)$ for any $n > 1$. The middle cohomology $H^{2(n-1)}(\mathbb{G}(1,n),\mathbb{Z})$ is generated by the cycles σ_{a_0,a_1} with $0 \leq a_0 \leq a_1 \leq n-1$ and $a_0 + a_1 = n-1$. In particular, for any (a_0,a_1) satisfying the above, the dual cycle corresponds to the sequence $((n-1)-a_1,(n-1)-a_0) = (a_0,a_1)$ because $a_0 + a_1 = n-1$. So every cycle is self-dual.

ii). Now look at $\mathbb{G}(n-2,n)$ for any $n > 1$. The middle cohomology $H^{(n-2)2}(\mathbb{G}(n-2,n),\mathbb{Z})$ is generated by cycles $\sigma_{a_0,\ldots,a_{n-2}}$ with $0 \leq a_0 \leq \cdots \leq a_{n-2} \leq 2$ and $a_0 + \cdots + a_{n-2} = n-2$. Clearly $(a_0,\ldots,a_{n-2}) = (1,\ldots,1)$ is a sequence satisfying the two criteria and the corresponding cycle is self-dual. Now, for each a_i equal to 2, there must be a corresponding a_j equal to 0 to keep the sum of a_i equal to $n-2$. Moreover, to satisfy monotonicity the strings of 0's and 2's must appear on the left and right ends of the sequence, respectively. It is clear that all cycles corresponding to such sequences are self-dual.

$\square$

3.3 Surfaces

The following result will be needed in one step of the classification of the smooth complete intersections with positive-definite intersection form

Definition 3.1. A smooth projective surface S is called a *fake* $\mathbb{P}^2$ if S is of general type (i.e. has Kodaira dimension equal to 2) and the Hodge diamond of S is equal to that of $\mathbb{P}^2$:

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 0 & & 1 & & 0 \\ & 0 & & 0 & \\ & & 1 & & \end{array}.$$

Theorem 3.1. *The only smooth projective surfaces with positive-definite intersection form are $\mathbb{P}^2$ and the fake $\mathbb{P}^2$ surfaces.*

We give a sketch of the proof, due to professor Michael Roth, which makes use of several results from surface theory (including the classification of minimal surfaces).

Proof Sketch. For a surface S, we recall the notation $p_g := h^0(K_S) = h^2(\mathcal{O}_S)$, $q := h^1(\mathcal{O}_S)$, and let $\kappa = \kappa(S)$ denote the Kodaira dimension. When S is smooth, its Hodge diamond then looks like

$$\begin{array}{ccccc} & & 1 & & \\ & q & & q & \\ p_g & & h^{1,1} & & p_g \\ & q & & q & \\ & & 1 & & \end{array}.$$

By Hodge index theorem, the index of the intersection form on S is $((2p_g+1)+, (h^{1,1}-1)-)$. So S has positive-definite intersection form only if $h^{1,1}(S) = 1$. In turn, the latter holds only if S is minimal, as $NS(S) = H^2(S,\mathbb{Z}) \cap H^{1,1}(S)$ by Lefschetz $(1,1)$. We check through the classification of minimal surfaces for potential examples:

$\kappa = -\infty$**:** Here we require $p_g = 0,\ q = 0$, so that the surface is $\mathbb{P}^2$.

$\kappa = 0$**: K3:** No examples, as $h^{1,1} = 20$.

Enriques: No examples, as $h^{1,1} = 10$.

Abelian surfaces: No examples, as $h^{1,1} = 4$.

Bi-elliptic: No examples, as $h^{1,1} \geq 2$.

$\kappa = 1$**:** These are the elliptic surfaces. We have $h^{1,1} \geq 2$, hence the are no examples among the elliptic surfaces.

$\kappa = 2$**:** These are surfaces of general type. Here we recall that

$$\tau = 2p_g + 2 - h^{1,1} = \frac{1}{3}(c_1^2 - 2\chi_{\mathrm{top}}) \quad \text{(Hirzebruch)}$$
$$1 - q + p_g = \frac{1}{12}(c_1^2 + \chi_{\mathrm{top}}) \quad \text{(Noether)},$$

where $c_1 := c_1(T_S) = -c_1(K_S)$ and χ_{top} is the topological Euler characteristic. If $h^{1,1} = 1$,

$$p_g + q = \frac{1}{3}(c_1^2 - 2\chi_{\mathrm{top}}) - \frac{1}{12}(c_1^2 + \chi_{\mathrm{top}}) = \frac{1}{4}(c_1^2 - 3\chi_{\mathrm{top}}) \leq 0 \quad \text{(Bogomolov – Miayoka – Yau)},$$

so that necessarily $p_g = q = 0$, so that the surface is a fake $\mathbb{P}^2$.

$\square$

Corollary 3.1. *No smooth projective surface with positive-definite intersection form is a complete intersection*

Proof. Suppose that S is a complete intersection of type $(d_1, \ldots, d_k)$ in $\mathbb{P}^{2+k}$. Let $H_1, \ldots, H_k$ denote the defining hypersurfaces. By adjunction formula, the canonical divisor K_S is equal to

$$K_S = \left(K_{\mathbb{P}^{2+k}} + H_1 + \cdots + H_k\right)\Big|_S$$
$$= ((-2-k-1) + d_1 + \cdots + d_k)H|_S \,,$$

where H is the hyperplane divisor in $\mathbb{P}^{2+k}$. Since we require $p_g = h^0(K_S) = 0$, we must also have $h^0(\mathcal{O}_{\mathbb{P}^{2+k}}((-k-3) + d_1 + \cdots + d_k)\Big|_S) = 0$. But then it is certainly necessary that

$$k + 3 > d_1 + \cdots + d_k,$$

as otherwise $h^0(\mathcal{O}_{\mathbb{P}^{2+k}}((-k-3) + d_1 + \cdots + d_k)) > 0$. The only possibilities for types are (2), (3) and $(2,2)$. The surface $X(2)$ can be realized as the image of the Segre embedding (up to a change of coordinates), so $h^{1,1}(X(2)) = 2$. The fact that $h^{1,1}(X(3)) \geq 2$ may be seen from the fact that $X(3)$ contains exactly 27 lines, with not all pairwise intersections of the lines equal (and, up to relabeling, the 27 lines on every smooth cubic have the same intersection matrix). Finally, by theorem 4.2 we have $h^{1,1}(X(2,2)) = 4$. $\square$

Chapter 4

Complete intersections with positive-definite intersection form, I

In this chapter we obtain a full list of smooth complete intersections with positive-definite intersection form. This is theorem 4.2. In the following chapter, we shall identify the lattices that occur.

4.1 Intersection form and the complex De Rham cohomology

Let X be a smooth projective variety of dimension n. Then X may be viewed as a compact Kähler manifold, with one choice of Kähler form coming from the Fubini-Study metric. We denote the intersection form on $H^n(X)$ by Q.

Extending scalars to $\mathbb{R}$ (using that $H^n(X,\mathbb{R}) = H^n(X)\otimes\mathbb{R}$ by the universal coefficient theorem), we obtain a real unimodular bilinear form on $H^n(X,\mathbb{R})$ agreeing with Q on the embedding $H^n(X) \hookrightarrow H^n(X,\mathbb{R})$. We denote the real extension of Q to $H^n(X,\mathbb{R})$ by Q again, by abuse of notation.

Now De Rham's theorem gives isomorphisms of real vector spaces $H^k_{DR}(X,\mathbb{R}) \cong H^k(X,\mathbb{R})$ for each $0 \leq k \leq 2n$. Together with the above paragraph, this fact suggests that the intersection form on $H^n(X)$ should have an interpretation from the point of view of calculus of differential forms on X. Such an interpretation is given by the following well-known lemma:

Lemma 4.1. *For $\alpha, \beta \in H^n(X, \mathbb{R})$, we have*

$$Q(\alpha, \beta) = \int_X \alpha \wedge \beta,$$

where on the right side we have identified the classes $\alpha, \beta \in H^n(X, \mathbb{R})$ with their images in $H^n_{DR}(X, \mathbb{R})$ under the De Rham isomorphism.

It will be helpful to further extend the bilinear form Q to the complex cohomology $H^n(X, \mathbb{C}) = H^n(X) \otimes \mathbb{C}$ (universal coefficient theorem) to make use of the Hodge and Lefschetz decompositions available on $H^n(X, \mathbb{C})$ in studying Q. Towards this end, letting ω denote a choice of Kähler form on X, we can define complex bilinear forms B_k on $H^k(X, \mathbb{C}) = H^k_{\mathrm{DR}}(X, \mathbb{C})$ for each $0 \leq k \leq n$ by

$$B_k(\alpha, \beta) := \int_X \omega^{n-k} \wedge \alpha \wedge \beta \ ,$$

where by abuse of notation ω denotes the class of the Kähler form in $H^2(X, \mathbb{C})$. The bilinear form B_k is well-defined on the level of cohomology by Stokes' theorem.

Because B_k is complex-valued, it does not make sense to ask whether it is definite. However, we make the following two observations: first, B_k is symmetric for k even and alternating for k odd; second, we have $\overline{B_k(\alpha, \beta)} = B_k(\overline{\alpha}, \overline{\beta})$, since ω is by definition real. For each $0 \leq k \leq n$, we may therefore define the Hermitian form

$$R_k(\alpha, \beta) := \begin{cases} B_k(\alpha, \overline{\beta}), & k \text{ even} \\ iB_k(\alpha, \overline{\beta}), & k \text{ odd} \end{cases}$$

on $H^k(X, \mathbb{C})$. Because the form R_k is Hermitian, it makes sense to ask whether R_k is positive-definite.

We make precise the relation between the forms Q and R_k in the case n even, $k = n$. By lemma 4.1, the restriction of the form R_n to $H^n(X,\mathbb{R})$ (the latter identified with its natural embedding in $H^n(X,\mathbb{C})$) agrees with the form Q. Therefore we may recover R_n from the values of Q on $H^n(X) \subset H^n(X,\mathbb{C})$ by extending Q sesquilinearly to $H^n(X,\mathbb{C})$. In particular, we have the following lemma:

Lemma 4.2. *Let $n = \dim X$ be even. Then the intersection form Q is positive-definite on $H^n(X)$ if and only if the Hermitian form R_n is positive-definite on $H^n(X,\mathbb{C})$.*

4.1.1 The form R_k on the Lefschetz decomposition of $H^k(X,\mathbb{C})$

The following two lemmas show that the form R_k is 'well-behaved' on the pieces $L^r H^{k-2r}_{\text{prim}}(X,\mathbb{C})$ of the Lefschetz decomposition of $H^k(X,\mathbb{C})$. In combination with the Hodge-Riemann bilinear relations, the two lemmas will be key in deriving necessary and sufficient conditions for the intersection form to be positive-definite in theorem 4.1.

Lemma 4.3. *The Lefschetz decomposition is orthogonal for the Hermitian form R_k. More precisely, if $\alpha \in L^r H^{k-2r}_{\text{prim}}(X,\mathbb{C})$ and $\beta \in L^s H^{k-2s}_{\text{prim}}(X,\mathbb{C})$ with $2r < 2s \leq k$, we have $R_k(\alpha,\beta) = 0$.*

Proof. Let $\alpha = \omega^r \wedge \alpha'$ and $\beta = \omega^s \wedge \beta'$, with $\alpha' \in H^{k-2r}_{\text{prim}}(X,\mathbb{C})$ and $\beta' \in H^{k-2s}_{\text{prim}}(X,\mathbb{C})$ primitive. We have $\omega^{n-k} \wedge \alpha \wedge \beta = (-1)^{(k-2r)(k-2s)} \left(\omega^{n-k+r+s} \wedge \alpha'\right) \wedge \beta'$. Since $r + s \geq 2r + 1$, we have $\omega^{n-k+r+s}\alpha' = 0$. Hence $R_k(\alpha,\beta) = 0$. □

Lemma 4.4. *The Hermitian form R_k induces the form R_{k-2r} on the $L^r H^{k-2r}_{\text{prim}}(X,\mathbb{C})$ part of the Lefschetz decomposition of $H^k(X,\mathbb{C})$. More precisely, for $\alpha, \beta \in L^r H^{k-2r}_{\text{prim}}(X,\mathbb{C}) \subset H^k(X,\mathbb{C})$, we have $R_k(\alpha,\beta) = R_{k-2r}(\alpha',\beta')$, where $\alpha = L^r\alpha'$ and $\beta = L^r\beta'$.*

Proof. The proof is a simple computation which we carry out for the case k even. In this case

$$\begin{aligned}
R_k(\alpha,\beta) &= B_k(L^r\alpha', \overline{L^r\beta'}) \\
&= B_k(L^r\alpha', L^r\overline{\beta'}) \quad \text{(since } \omega \text{ is real)} \\
&= \int_X \omega^{n-k} \wedge (\omega^r \wedge \alpha') \wedge (\omega^r \wedge \overline{\beta'}) \\
&= (-1)^{(k-2r)2r} \int_X \omega^{n-(k-2r)} \wedge \alpha' \wedge \overline{\beta'} \\
&= R_{k-2r}(\alpha', \beta'),
\end{aligned}$$

since $(k-2r)2r$ is even. □

4.1.2 Hodge-Riemann bilinear relations

Let k be even. We write down the Hodge-Riemann relations identifying the parts of the Hodge decomposition of $H^k_{\text{prim}}(X,\mathbb{C})$ on which the Hermitian form $R_k(\alpha,\beta)$ is positive- or negative-definite.

Theorem 4.1. *Let $p+q=k$. For any $\alpha \in H^{p,q}_{\text{prim}}(X,\mathbb{C})$, we have*

$$(-1)^{(p-q+k(k-1))/2} R_k(\alpha,\alpha) \geq 0$$

with equality if and only if $\alpha = 0$.

We observe that, since $p+q=k$ is even, $p-q=(p+q)-2q$ is also even. Also, $k(k-1)$ is clearly even, since one of k and $k-1$ is even. Therefore $(p-q+k(k-1))/2$ is an integer. For a proof of the relations, invoking the 'Kähler identities', please see [14] pp. 150–153.

Remark. Similar relations hold in the case k odd—see [14] pp. 152–153.

The Hodge-Riemann relations imply the following useful characterization of the pieces of the Hodge decomposition of the middle cohomology on which the intersection form is positive-definite:

Proposition 4.1. *Let $n = \dim X$ be even. Then the intersection form on $H^n(X)$ is positive-definite if and only if $h^{p,q}_{\mathrm{prim}}(X) = 0$ for all p, q with $p \cdot q$ odd.*

Proof. By lemma 4.2, it is equivalent to show that the Hermitian form R_n on $H^n(X, \mathbb{C})$ is positive-definite if and only if $h^{p,q}_{\mathrm{prim}}(X) = 0$ for all p, q with $p \cdot q$ odd. Motivated by the Lefschetz decomposition of the middle cohomology

$$H^n(X, \mathbb{C}) = \bigoplus_{2r \leq n} L^r H^{n-2r}_{\mathrm{prim}}(X),$$

and lemmas 4.4 and 4.3, we begin by finding necessary and sufficient conditions for the Hermitian form R_{2k} to be positive-definite on $H^{n-2k}_{\mathrm{prim}}(X, \mathbb{C})$. Towards this end, we have the following claim:

Claim. *For each $0 \leq 2k \leq n$, the Hermitian form R_{2k} on $H^{2k}_{\mathrm{prim}}(X, \mathbb{C})$ is positive-definite if and only if $h^{p,q}_{\mathrm{prim}}(X) = 0$ for all p, q with $p + q = 2k$ and $p \cdot q$ odd.*

Proof of claim. Let $\alpha \in H^{p,q}_{\mathrm{prim}}(X)$, with $p + q = 2k$. The Hodge-Riemann relation for R_{2k} on $H^{p,q}_{\mathrm{prim}}(X, \mathbb{C})$ is

$$(-1)^{(p-q+2k(2k-1))/2} R_{2k}(\alpha, \alpha) \geq 0,$$

with equality if and only if $\alpha = 0$.

Suppose first that the Hermitian form R_{2k} is positive-definite. Then by above $h^{p,q}_{\mathrm{prim}}(X) = 0$ whenever $(p - q + 2k(2k-1))/2$ is odd. The latter implies that

$$\begin{aligned} p - q + 2k(2k-1) \equiv 2 \mod 4 \quad &\Longrightarrow \quad p - q + 4k^2 - 2k \equiv 2 \mod 4 \\ &\Longrightarrow \quad p - q - (p+q) \equiv 2 \mod 4 \\ &\Longrightarrow \quad -2q \equiv 2 \mod 4, \end{aligned}$$

which shows that q is odd. Since $q = k - p$ and k is even, it follows that p is odd, hence $p \cdot q$ is odd. To see the converse, simply note that each of the steps of the argument above was reversible. $\square$

We now apply the claim to each component $L^k H^{n-2k}_{\text{prim}}(X, \mathbb{C})$ of the Lefschetz decomposition. Using the observation that the form R_n induces the form R_{n-2k} on this component (lemma 4.4), we find that R_n is positive definite on the component $L^k H^{n-2k}_{\text{prim}}(X, \mathbb{C})$ if and only if $h^{p,q}_{\text{prim}}(X) = 0$ for all p, q with $p + q = n - 2k$ and $p \cdot q$ odd. Then by orthogonality of the Lefschetz decomposition for R_n (lemma 4.3), we have that R_n is positive-definite on $H^n(X, \mathbb{C})$ if and only if $h^{p,q}_{\text{prim}}(X) = 0$ for all p, q with $p + q = n - 2k$ for some $2k \leq n$ and $p \cdot q$ odd. Finally, by Serre duality R_n is positive-definite on $H^n(X, \mathbb{C})$ if and only if $h^{p,q}_{\text{prim}}(X) = 0$ for all p, q.

□

It follows that a smooth projective variety X has a positive-definite intersection form on its middle (integral) cohomology only if $h^{1,1}_{\text{prim}}(X) = 0$, hence only if $h^{1,1}(X) = h^{0,0}(X) = 1$. By the Lefschetz theorem on (1,1) classes, $NS(X) = H^{1,1}(X) \cap H^2(X, \mathbb{Z})$ (here $H^2(X, \mathbb{Z})$ is identified with its natural image in $H^2(X, \mathbb{C})$), whence necessarily $\text{rk}_{\mathbb{Z}} NS(X) = 1$. We have shown:

Corollary 4.1. *A smooth projective variety with positive-definite intersection form on its middle cohomology necessarily has Picard number one.*

4.2 Which complete intersections have a positive-definite intersection form?

This section is devoted to classifying the types $\mathbf{d} = (d_1, \ldots, d_r)$ of the smooth complete intersections $X(\mathbf{d})$ in $\mathbb{P}^{n+r}$ whose intersection form on the middle cohomology $H^n(X(\mathbf{d}))$ is positive-definite. We have

Theorem 4.2. *The smooth complete intersections $X(\mathbf{d})$ with positive-definite intersection form on their middle cohomology are exactly those of the following type and dimension:*

$\mathbf{d}$	$n := \dim X(\mathbf{d})$	$\operatorname{rank} H^n(X(\mathbf{d}))$
(2)	$4s\ (s \geq 1)$	2
$(2,2)$	$4s\ (s \geq 1)$	$4(s+1)$

As part of the proof, we shall also show the following statement, useful in chapter 5.

Lemma 4.5. *The rank of the middle cohomology of a smooth quadric hypersurface in $\mathbb{P}^{2k+1}$ (k a positive integer) is 2.*

4.2.1 Proof of theorem 4.2

Let $X(\mathbf{d})$ be a smooth complete intersection of dimension n and type $\mathbf{d} = (d_1, \ldots, d_r)$ in $\mathbb{P}^{n+r}$ and suppose X has a positive-definite intersection form. We may assume that n is even, say $n = 2m$, as the cup product is antisymmetric for n odd. Moreover, because no smooth projective surface has a positive-definite intersection form (corollary 3.1), we may assume $m > 1$. This is the one step that uses the classification of smooth projective surfaces with positive-definite intersection form in Chapter 3.

We now use the following theorem to see that the only part of the integral cohomology of a smooth complete intersection that doesn't coincide with the cohomology of projective space of the same dimension is the middle cohomology.

Theorem 4.3 (Lefschetz theorem on hyperplane sections)**.** *Let $X \subset \mathbb{P}^N$ be an n-dimensional projective variety (not necessarily smooth) with a choice of embedding in $\mathbb{P}^N$ and $Y = \mathbb{P}^{N-1} \cap X$ a hyperplane section such that $U := X \smallsetminus Y$ is smooth and n-dimensional. Then the morphism $H^k(X, \mathbb{Z}) \to H^k(Y, \mathbb{Z})$ induced by the inclusion $Y \hookrightarrow X$ is an isomorphism for $k < n-1$ and is injective for $k = n-1$.*

Proof. See [15], pp. 33–34. $\square$

Corollary 4.2. *Let $X = X(\mathbf{d})$ be a smooth complete intersection in $\mathbb{P}^{n+r}$, with $\mathbf{d} = (d_1, \ldots, d_r)$. Then for all $i \neq n$,*

$$H^i(X, \mathbb{Z}) = \begin{cases} \mathbb{Z}\ \sigma^i, & i \text{ even} \\ 0, & i \text{ odd} \end{cases}$$

where σ is defined as the hyperplane section $X \cap \mathbb{P}^{n+r-1}$ for a general $\mathbb{P}^{n+r-1}$ (equivalently, σ is the image of the hyperplane class $H \in H^2(\mathbb{P}^{n+r}, \mathbb{Z})$ under the map $H^2(\mathbb{P}^{n+r}, \mathbb{Z}) \to H^2(X, \mathbb{Z})$ induced by the inclusion).

Proof. It is enough to show the theorem for $r = 1$. Let X be a hypersurface of degree d in $\mathbb{P}^{n+1}$. Let $v : \mathbb{P}^{n+1} \to \mathbb{P}^N$ be the degree d Veronese embedding (where we have abused notation by omitting the subscript on v_d), with $N = \binom{n+1+d}{d} - 1$. Then $v(X)$ is a hyperplane section of the image of $\mathbb{P}^{n+1}$ in $\mathbb{P}^N$, hence we may apply the theorem on hyperplane sections to conclude that the rational cohomology of $v(X)$ agrees with that of $v(\mathbb{P}^{n+1})$ outside of $H^n(X, \mathbb{Q})$. Since v is biregular, in particular $H^i(\mathbb{P}^{n+1}, \mathbb{Z}) \cong H^i(v(\mathbb{P}^{n+1}), \mathbb{Z})$ and $H^i(X, \mathbb{Z}) \cong H^i(v(X), \mathbb{Z})$ for all $0 \leq i \leq 2(n+1)$, which shows the claim. □

So the only interesting part of the integral cohomology of X is $H^n(X)$. In particular, $\sigma \in H^{1,1}(X)$ because it is the cohomology class of a cycle, so we have $h^{p,q}_{\text{prim}}(X) = 1 - 1 = 0$ for all p, q with $p + q \neq n$. By proposition 4.1, we further have $h^{p,q}_{\text{prim}}(X) = 0$ for $p \cdot q$ odd and $p + q = n$. We wish to show that the last condition turns out to place a strict restriction on the Hodge decomposition of $H^n(X, \mathbb{C})$, which does not hold for most possibilities for $\mathbf{d}$.

We begin by defining the following generating polynomial. Let X be a smooth projective variety (not necessarily a complete intersection) and set

$$\chi_y(X) := \sum_{p=0}^{n} \left(\sum_{q=0}^{n} (-1)^q h^{p,q}(X) \right) y^p$$

We observe that substituting $y = -1$ in $\chi_y(X)$ recovers the topological Euler characteristic of X. Similarly, in the case n even, substituting $y = 1$ recovers the signature of the symmetric form B_n on $H^n(X, \mathbb{R})$ (this is one version of the Hodge Index theorem, as it appears in, say, [14] Theorem 6.33). Substituting $y = 0$ gives the Euler characteristic of the structure sheaf of X.

In case $X = X(\mathbf{d})$ is a complete intersection, we have the following

Theorem 4.4 (Hirzebruch). *If X is a complete intersection of type $(d_1, \ldots, d_r)$ in $\mathbb{P}^{n+r}$, then $\chi_y(X)$ is equal to the coefficient of z^{n+r} in*

$$\frac{1}{(1-z)(1+yz)} \prod_{i=1}^{r} \frac{(1+yz)^{d_i} - (1-z)^{d_i}}{(1+yz)^{d_i} + y(1-z)^{d_i}}.$$

Proof. See [8], pg. 160. The proof relies on the Hirzebruch-Riemann-Roch theorem. □

Using theorem 4.4 and elementary manipulations with formal power series Deligne showed the following:

Theorem 4.5 (Deligne). *With notations as above,*

a). Let $p \leq p' \leq q' \leq q$ with $p + q = p' + q'$. Then

$$h^{p,q}_{\text{prim}}(X(\mathbf{d}), \mathbb{C}) \leq h^{p',q'}_{\text{prim}}(X(\mathbf{d}), \mathbb{C}).$$

b). For $q \leq p$, $h^{p,q}_{\text{prim}}(\mathbf{d}) \neq 0$ if and only if

$$q > \frac{p + q + r - \sum d_i}{d_r}.$$

Proof. See [3] pp. 54 – 58. □

The two parts of theorem 4.5 motivate the following definition:

Definition 4.1. Let X be a smooth projective variety of dimension n. Then X is said to have *coniveau c* (sometimes also called *Hodge level c*) if c is the lowest integer such that $h^{n-c,c}_{\text{prim}}(X(\mathbf{d}), \mathbb{C}) \neq 0$.

Remark. Since $h_{\text{prim}}^{n-c,c}(X,\mathbb{C}) = h_{\text{prim}}^{c,n-c}(X,\mathbb{C})$ by symmetry under complex conjugation, X has coniveau $\leq \lfloor \frac{n}{2} \rfloor$.

Remark. Theorem 4.5-a does not hold for general smooth projective varieties. For example, a rigid Calabi-Yau threefold has the Hodge diamond

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & h^{1,1} & & 0 & \\ 1 & & 0 & & 0 & & 1 \\ & 0 & & h^{1,1} & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

(see [2], pg.517).

Finally, we restate theorem 4.5 for the middle cohomology of a smooth complete intersection in the following convenient form:

Theorem 4.6. *Let* $\mathbf{d} = (d_1, \dots, d_r)$. *Then* $X(\mathbf{d}) \subset \mathbb{P}^{n+r}$ *has coniveau* $\geq c$ *if and only if*

$$n + r \geq \sum_{i=1}^{r} d_i + (c-1)d_r.$$

We now proceed to the proof of the classification. We consider two cases according to whether $m = (\dim X)/2$ is odd or even:

- *The case* $m \equiv 0 \mod 2$: The Hodge decomposition of the middle primitive cohomology $H_{\text{prim}}^n(X,\mathbb{C})$ looks like

$$H_{\text{prim}}^{n,0}(X) \oplus \cdots \oplus H_{\text{prim}}^{m+1,m-1}(X) \oplus H_{\text{prim}}^{m,m}(X) \oplus H_{\text{prim}}^{m-1,m+1}(X) \oplus \cdots \oplus H_{\text{prim}}^{0,n}(X).$$

  Applying proposition 4.1, we have $h_{\text{prim}}^{m+1,m-1}(X) = h_{\text{prim}}^{m-1,m+1}(X) = 0$, hence by theorem 4.5 all parts of the Hodge decomposition except possibly $H_{\text{prim}}^{m,m}(X)$ are zero.

Therefore X is of coniveau $\geq m$. By theorem 4.6, we have

$$2m + r \geq \sum_{i=1}^{r} d_i + (m-1)d_r, \tag{4.1}$$

which may be rearranged to

$$r \geq \sum_{i=1}^{r-1} d_i + m(d_r - 2) \geq 2(r-1).$$

Now $r \geq 2(r-1)$ is quickly seen to be equivalent to $r \leq 2$, so the only possibilities for r are $r = 1$ and $r = 2$, which we investigate in turn.

- $r = 1$: We have

  $$2m + 1 \geq m \cdot d_1 \quad \iff \quad d_1 \leq 2 + \frac{1}{m} < 3 \quad \text{for } m > 1.$$

  The only possibility for $\mathbf{d}$ is $(d_1) = (2)$ (with no additional restrictions on m).

- $r = 2$: We have

  $$2(m+1) \geq d_1 + md_2 \geq 2(m+1).$$

  The only possibility for $\mathbf{d}$ is $(d_1, d_2) = (2, 2)$ (with no additional restrictions on m).

It is immediate to check that both $\mathbf{d} = (2)$ and $\mathbf{d} = (2, 2)$ satisfy the inequality (4.1) with no additional restrictions on m.

- *The case* $m \equiv 1 \mod 2$: Proposition 4.1 implies that $h^{m,m}_{\text{prim}}(X) = 0$, hence $h^{m,m}(X) = h^{m-1,m-1}(X) = 1$. The Hodge decomposition of the primitive middle cohomology $H^n_{\text{prim}}(X, \mathbb{C})$ looks like

$$\cdots \oplus H^{m+2,m-2}_{\text{prim}}(X) \oplus H^{m+1,m-1}_{\text{prim}}(X) \oplus H^{m,m}_{\text{prim}}(X) \oplus H^{m-1,m+1}_{\text{prim}}(X) \oplus H^{m-2,m+2}_{\text{prim}}(X) \oplus \cdots$$

Arguing as in the previous case, proposition 4.1 and theorem 4.5 imply that X is of coniveau $\geq (m-1)$. We investigate the restrictions that the coniveau condition imposes on $\mathbf{d}$ and show that for all possibilities for $\mathbf{d}$ and m, $h^{m,m}(X) \neq 1$. By theorem 4.6,

$$2m + r \geq \sum_{i=1}^{r} d_i + (m-2)d_r,$$

which may be rearranged to

$$r \geq \sum_{i=1}^{r-1} d_i + d_r(m-1) - 2m \geq 2(r-1) + 2(m-1) - 2m = 2(r-2).$$

Now, $r \geq 2(r-2)$ is equivalent to $r \leq 4$. We investigate the cases $r = 1, 2, 3, 4$ in turn.

- $r = 1$: We have

$$2m + 1 \geq (m-1)d_1 \qquad \Longleftrightarrow \qquad d_1 \leq 2 + \frac{3}{m-1}.$$

  The possibilities for $\mathbf{d}$ are

  - $\mathbf{d} = (d_1) = (2)$ (with no additional restrictions on m)
  - $\mathbf{d} = (d_1) = (3)$, $m = 3$. (A cubic in $\mathbb{P}^7$).

- $r = 2$: We have

$$2(m+1) \geq d_1 + (m-1)d_2 \geq 2 + (m-1)d_2.$$

  Rearranging,

$$d_2 \leq \frac{2m}{m-1} = 2 + \frac{2}{m-1}.$$

  The possibilities for $\mathbf{d}$ are

  - $\mathbf{d} = (d_1, d_2) = (2, 2)$ (with no additional restrictions on m)
  - $\mathbf{d} = (d_1, d_2) = (2, 3)$, $m = 3$. (Complete intersection of a quadric and a cubic in $\mathbb{P}^8$).

(In the case $\mathbf{d} = (3,3)$, the original inequality is not satisfied for $m > 1$).

- $r = 3$: We have

$$2m + 3 \geq d_1 + d_2 + (m-1)d_3 \geq 4 + (m-1)d_3.$$

  Rearranging,

$$d_3 \leq \frac{2m-1}{m-1} = 2 + \frac{1}{m-1} < 3 \quad \text{for } m > 1.$$

  The only possibility for $\mathbf{d}$ is $(d_1, d_2, d_3) = (2,2,2)$ with no additional restrictions on m.

- $r = 4$: We have

$$2m + 4 \geq d_1 + d_2 + d_3 + (m-1)d_4 \geq 2m + 4.$$

  The only possibility for $\mathbf{d}$ is $(d_1, d_2, d_3, d_4) = (2,2,2,2)$ with no additional restrictions on m.

Summarizing, the possibilities for $\mathbf{d}$ in the case $m \equiv 1 \mod 2$ are the 'families' of complete intersections of one, two, three and four quadrics with no additional restrictions on m, as well as the 'sporadic cases' of a cubic in $\mathbb{P}^7$ and a complete intersection of a quadric and a cubic in $\mathbb{P}^8$. We show that $h^{m,m}(X) \neq 1$ in all of the above possibilities.

To find $h^{m,m}(X)$, we observe that in the case $X = X(\mathbf{d})$ of dimension $n = 2m$, we have

$$(-1)^m h^{m,m}(X) = \sum_{q=0}^{n} (-1)^q h^{m,q}(X),$$

as all terms other than $h^{m,m}(X)$ in the right-hand sum are zero (this may be seen from, say, the description of the Hodge diamond of X obtained by applying the Lefschetz theorem on hyperplane sections). We now apply Hirzebruch's generating series: the coefficient of $y^m z^{2m+r}$ of

$$\frac{1}{(1-z)(1+yz)}\prod_{i=1}^{r}\frac{(1+yz)^{d_i}-(1-z)^{d_i}}{(1+yz)^{d_i}+y(1-z)^{d_i}}$$

is $(-1)^m h^{m,m}(X)$. The observation that only the terms of the form $y^m z^{2m+r}$ are relevant somewhat simplifies the computations that follow, as some of the intermediate terms may be omitted.

We begin with a preliminary computation useful for dealing with the families of quadrics. Each term of the product $\prod_{i=1}^{r}\frac{(1+yz)^{d_i}-(1-z)^{d_i}}{(1+yz)^{d_i}+y(1-z)^{d_i}}$ in these cases looks like

$$\begin{aligned}\frac{(1+yz)^2-(1-z)^2}{(1+yz)^2+y(1-z)^2} &= \frac{(1+2yz+y^2z^2)-(1-2z+z^2)}{(1+2yz+y^2z^2)+y(1-2z+z^2)}\\ &= \frac{2z(1+y)+z^2(y^2-1)}{1+y+yz^2(1+y)}\\ &= \frac{(1+y)z(2+z(y-1))}{(1+y)(1+yz^2)}\\ &= z\cdot\frac{(1+yz)+(1-z)}{1+yz^2}\end{aligned}$$

Now, the coefficient of $y^m z^{2m+r}$ in

$$\frac{1}{(1-z)(1+yz)}\left(\frac{(1+yz)^2-(1-z)^2}{(1+yz)^2+y(1-z)^2}\right)^r = \frac{1}{(1-z)(1+yz)}z^r\left(\frac{((1+yz)+(1-z))}{1+yz^2}\right)^r$$

is the coefficient of $y^m z^{2m} = (yz^2)^m$ in

$$\frac{1}{(1-z)(1+yz)}\left(\frac{(1+yz)+(1-z)}{1+yz^2}\right)^r.$$

Motivated by this observation, somewhat informally we let $\mathcal{O}$ stand for the terms not of the form $(yz^2)^m$ for some $m \geq 0$ in $\mathbb{C}[[y,z]]$, the ring of formal power series in y and z with coefficients in $\mathbb{C}$. Thus for any $f(y,z) \in \mathbb{C}[[y,z]]$, we can write

$$f(y,z) = \sum_{d=0}^{\infty} a_d (yz^2)^d + \mathcal{O}.$$

Finally, we have the useful identity

$$\frac{1}{(1+yz^2)^r} = \sum_{d=0}^{\infty}(-1)^d\binom{d+r-1}{r-1}(yz^2)^d$$

for all $r \geq 1$.

- *The case* $\mathbf{d} = (2)$, $m \equiv 1 \mod 2$*:* We have

$$\begin{aligned}\frac{1}{(1-z)(1+yz)}\left(\frac{(1+yz)+(1-z)}{1+yz^2}\right) &= \sum_{d=0}^{\infty}(-1)^d(yz^2)^d\cdot\left(\sum_{j=0}^{\infty}z^j+\sum_{k=0}^{\infty}(-1)^k(yz)^k\right)\\ &= \sum_{d=0}^{\infty}(-1)^d(yz^2)^d\cdot(1+\mathcal{O}+1+\mathcal{O})\\ &= \sum_{d=0}^{\infty}(-1)^d2(yz^2)^d+\mathcal{O}.\end{aligned}$$

  The coefficient of $(yz^2)^m$ is $(-1)^m2$, hence $h^{m,m}(X(2)) = 2 \neq 1$ for all $m > 0$; in particular, for all $m \equiv 1 \mod 2$.

Proof of lemma 4.5. We note that in the case of a smooth quadric hypersurface in $\mathbb{P}^{2k+1}$, theorem 4.6 becomes the following statement: $X(2)$ has coniveau $\geq c$ if and only if $2k+1 \geq 2c$. Since also $c \leq k$, it follows that the only nonzero term of the Hodge decomposition of the middle cohomology of a smooth quadric is $h^{k,k}(X(2))$, which the computation above shows is equal to 2 for all k. Hence also the rank of $H^{2k}(X(2))$ is equal to 2 for all k. This proves lemma 4.5. □

Remark. Lemma 4.5 will be useful in determining the cup product lattice on the middle cohomology of $X(2)$ in section 5.1. In the case $k = 1$, the fact that the middle cohomology of a smooth quadric in $\mathbb{P}^3$ has rank two is classical: such a surface may be realized as the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^1$ in $\mathbb{P}^3$ and is in particular doubly-ruled.

- *The case* $\mathbf{d} = (2,2)$, $m \equiv 1 \mod 2$*:* We have

$$\begin{aligned}\frac{1}{(1-z)(1+yz)}\left(\frac{(1+yz)+(1-z)}{1+yz^2}\right)^2 &= \frac{1}{(1+yz^2)^2}\frac{(1-z)^2+2(1-z)(1+yz)+(1+yz)^2}{(1-z)(1+yz)} \\ &= \frac{1}{(1+yz^2)^2}\left(\frac{1-z}{1+yz}+2+\frac{1+yz}{1-z}\right) \\ &=: \frac{1}{(1+yz^2)^2}\left(A+2+B\right).\end{aligned}$$

Now,

$$A = (1-z)\sum_{j=0}^{\infty}(-1)^j(yz)^j = 1+yz^2+\mathcal{O},$$

$$B = (1+yz)\sum_{j=0}^{\infty}z^j = 1+yz^2+\mathcal{O}$$

and so

$$A+2+B = 2(2+yz^2)+\mathcal{O}.$$

We also have

$$\frac{1}{(1+yz^2)^2} = \sum_{d=0}^{\infty}(-1)^d\binom{d+1}{1}(yz^2)^d = \sum_{d=0}^{\infty}(-1)^d(d+1)(yz^2)^d.$$

Hence,

$$\begin{aligned}\frac{1}{(1+yz^2)^2}\left(A+2+B\right) &= 2\sum_{d=0}^{\infty}(-1)^d(d+1)(yz^2)^d\cdot\left(2+yz^2+\mathcal{O}\right) \\ &= 2\left(\sum_{d=0}^{\infty}\left(2(-1)^d(d+1)+(-1)^{d-1}d\right)(yz^2)^d\right)+\mathcal{O}\end{aligned}$$

The coefficient of $(yz^2)^m$ is $(-1)^m\,2(m+2)$, hence $h^{m,m}(X(2,2)) = 2(m+2) > 1$ for all $m \equiv 1 \mod 2$.

- *The case* $\mathbf{d} = (2,2,2)$, $m \equiv 1 \mod 2$*:* We have

$$\begin{aligned}
\frac{1}{(1-z)(1+yz)}\left(\frac{(1+yz)+(1-z)}{1+yz^2}\right)^3 &= \frac{(1-z)^3+(1+yz)^3+3(1-z)(1+yz)(1-z+1+yz)}{(1+yz^2)^3(1-z)(1+yz)} \\
&= \frac{1}{(1+yz^2)^3}\left(\frac{(1-z)^2}{1+yz}+\frac{(1+yz)^2}{1-z}+3(2+yz-z)\right) \\
&=: \frac{1}{(1+yz^2)^3}\left(C+D+6+3yz-3z\right) \\
&= \frac{1}{(1+yz^2)^3}\left(C+D+6+\mathcal{O}\right).
\end{aligned}$$

Now,

$$C = (1-2z+z^2)\sum_{j=0}^{\infty}(-1)^j(yz)^j = 1+2yz^2+y^2z^4+\mathcal{O},$$

$$D = (1+2yz+y^2z^2)\sum_{j=0}^{\infty}z^j = 1+2yz^2+y^2z^4+\mathcal{O}$$

and so

$$C+D+6 = 2(4+2yz^2+y^2z^4)+\mathcal{O}.$$

We also have

$$\frac{1}{(1+yz^2)^3} = \sum_{d=0}^{\infty}(-1)^d\binom{d+2}{2}(yz^2)^d.$$

Hence,

$$\begin{aligned}
\frac{1}{(1+yz^2)^3}\left(C+D+6+\mathcal{O}\right) &= \sum_{d=0}^{\infty}(-1)^d\binom{d+2}{2}(yz^2)^d\cdot(2(4+2yz^2+y^2z^4)+\mathcal{O}) \\
&= 2\left(\sum_{d=0}^{\infty}\left(4(-1)^d\binom{d+2}{2}+2(-1)^{d-1}\binom{d-1}{2}+(-1)^{d-2}\binom{d}{2}\right)(yz^2)^d\right)+\mathcal{O}
\end{aligned}$$

The coefficient of $(yz^2)^m$ is $(-1)^m(3m(m+3)+8)$, hence $h^{m,m}(X(2,2,2)) = 3m^2+9m+8 > 1$ for all $m \equiv 1 \mod 2$.

- *The case* $\mathbf{d} = (2,2,2,2)$, $m \equiv 1 \mod 2$*:* Similarly to the above three cases, one computes $h^{m,m}(X(2,2,2,2)) = 2\left(8\binom{m+3}{3}+3\binom{m+1}{1}+\binom{m}{3}\right) = 10m^3/3+17m^2+83m/3+16 > 1$ for all $m \equiv 1 \mod 2$.

- *The case* $\mathbf{d} = (3)$, $m = 3$*:* One computes that the coefficient of $y^3 z^7$ in

$$\frac{1}{(1-z)(1+yz)} \frac{(1+yz)^3 - (1-z)^3}{(1+yz)^3 + y(1-z)^3}$$

  is 71. So $h^{3,3}(X(3)) = 71 \neq 1$.

- *The case* $\mathbf{d} = (2,3)$, $m = 3$: One computes that the coefficient of $y^3 z^8$ in

$$\frac{1}{(1-z)(1+yz)} \frac{(1+yz)^2 - (1-z)^2}{(1+yz)^2 + y(1-z)^2} \frac{(1+yz)^3 - (1-z)^3}{(1+yz)^3 + y(1-z)^3}$$

  is 252. So $h^{3,3}(X(2,3)) = 252 \neq 1$.

Chapter 5

Complete intersections with positive-definite intersection form, II

Having obtained a complete list of the types of smooth complete intersections with positive-definite intersection form in theorem 4.2, we now classify the lattices that appear as the cup product lattices on the middle cohomology of the varieties in the list. The middle cohomology of an even-dimensional quadric has rank two by lemma 4.5, with the cup product lattice alternating between the identity lattice and the hyperbolic plane lattice as the dimension is increased by two (in particular, positive-definite in dimensions divisible by four). The cup product lattice of a smooth complete intersection of two quadrics in $\mathbb{P}^{4k+2}$ is $\Gamma_{4(k+1)}$. In particular, we have the wonderful result that the cup product lattice of $X(2,2) \subset \mathbb{P}^6$ is E_8.

5.1 Special forms of equations defining quadrics and complete intersections of two quadrics

Lemma 5.1. *a) The quadric hypersurface in $\mathbb{P}^n$ defined as the vanishing locus of the form $Q = \sum_{i=0}^{n} Z_i^2$ (where Z_i are coordinates of $\mathbb{P}^n$) is smooth.*

b) Up to a change of coordinates, every smooth quadric hypersurface in $\mathbb{P}^n$ is the vanishing locus of the form Q in a).

c) If Q and Q' are the forms

$$Q = \sum_{i=0}^{n} Z_i^2 \quad \text{and} \quad Q' = \sum_{i=0}^{n} a_i Z_i^2 \quad \text{where } a_i \neq a_j \text{ for } i \neq j$$

then the intersection $\{Q = 0\} \cap \{Q' = 0\}$ in $\mathbb{P}^n$ is smooth of codimension 2.

d) Up to a change of coordinates, every smooth complete intersection of type (2,2) in $\mathbb{P}^n$ is the intersection of hyperplanes defined by the forms Q and Q' in c).

Proof of part d). Proceed by induction on $n = \dim(\mathbb{P}^n)$. The base case is two smooth conics in $\mathbb{P}^2$ intersecting in four points. A smooth conic in $\mathbb{P}^2$ intersects any line in two points (in particular, contains no lines), so the four points are in general position. So there is a $\mathrm{PGL}(3, \mathbb{C})$ action taking the four points to the points

$$[1 : \sqrt{2}i : 1], \; [-1 : \sqrt{2}i : 1], \; [1 : -\sqrt{2}i : 1], \; [-1 : -\sqrt{2}i : 1],$$

which are the intersection of the quadrics

$$Z_0^2 + Z_1^2 + Z_2^2 = 0,$$
$$Z_0^2 + 2Z_1^2 + 3Z_2^2 = 0.$$

Now proceed to the general case. Let X be a smooth complete intersection of type $(2,2)$ in $\mathbb{P}^n$. First, we show

Claim. *We may assume that the definining quadrics Q_1 and Q_2 of X have full rank (therefore are smooth and their associated symmetric forms nondegenerate).*

Proof. Look at the pencil $\lambda Q_1 + \mu Q_2$, $[\mu : \lambda] \in \mathbb{P}^1$. By Bertini's theorem, singularities of a general member of the pencil can occur only on the base points of the linear series, which is X. Since X is smooth, a general member of the pencil is smooth, so up to a change of generators Q_1 and Q_2 have full rank. $\square$

Let $\langle\cdot,\cdot\rangle_Q$ denote the symmetric biliinear form associated to a quadric Q.

We begin by showing that to set up the induction on dimension, it is enough to find a hyperplane H and $v \in \mathbb{P}^n \smallsetminus X$ such that $H = v^{\perp}_{Q_1} = v^{\perp}_{Q_2}$, where $v^{\perp}_{Q_j} := \{w \in \mathbb{P}^n : \langle v, w\rangle_{Q_j} = 0\}$ for $j = 1, 2$. Indeed, assuming existence of v and H, choose a basis $w_0, \dots, w_n$ of $\mathbb{C}^{n+1}$ so that $v = [1 : 0 : \cdots : 0]$ and $H = \mathbb{P}(\operatorname{span}(w_1, \dots, w_n))$, so that $H = \{Z_0 = 0\}$. We may further scale w_0 so that $\langle v, v\rangle_{Q_1} = 1$. With respect to this basis, forms Q_1 and Q_2 look like

$$Q_1 = Z_0^2 + \sum_{1\le i,j} b_{ij} Z_i Z_j \quad \text{and} \quad Q_2 = a_0 Z_0^2 + \sum_{1\le i,j} c_{ij} Z_i Z_j.$$

Now $X \cap H$ is a complete intersection of the two quadrics $Q'_j := Q_j|_H$, $j = 1, 2$ (with the choice of coordinates above, identifying H with $\{Z_0 = 0\}$, we have $Q'_j = Q_j([0 : Z_1 : \cdots : Z_n])$ for $j = 1, 2$). Moreover, $X \cap H$ has dimension $n - 1$. We check that $X \cap H$ is smooth if X is smooth, so that we're done by induction. The matrix of partial derivatives of Q_1 and Q_2 looks like

$$J(Z_0, \dots, Z_n) = \begin{pmatrix} 2Z_0 & A_{11} & \cdots & A_{1n} \\ 2a_0 Z_0 & A_{21} & \cdots & A_{2n} \end{pmatrix}$$

where the $A_{ij} = A_{ij}(Z_1, \dots, Z_n)$ (in particular, the terms A_{ij} don't involve Z_0). On the other hand, the matrix of partial derivatives of Q'_1 and Q'_2 is simply

$$J'(Z_1, \dots, Z_n) = \begin{pmatrix} A_{11} & \cdots & A_{1n} \\ A_{21} & \cdots & A_{2n} \end{pmatrix}$$

But, identifying H with $\{Z_0 = 0\}$, if $p \in X \cap H$, then $p = [0 : Z_1 : \cdots : Z_n]$, so that

$$J(p) = \begin{pmatrix} 0 & A_{11} & \cdots & A_{1n} \\ 0 & A_{21} & \cdots & A_{2n} \end{pmatrix}$$

has rank equal to the rank of $J'(Z_1, \ldots, Z_n)$. Since X is smooth $J(p)$ has full rank at all points $p \in X$ and it follows that $X \cap H$ is smooth.

It remains to show that we can find the desired point v. Let Q_1 and Q_2 continue denoting a choice of nondegenerate quadrics cutting out X. By nondegeneracy, Q_j induces an isomorphism

$$\begin{aligned} \mathbb{P}^n &\xrightarrow{\psi} \mathbb{P}^{n*} \\ v &\longmapsto v^{\perp}_{Q_j} = \{w \in \mathbb{P}^n : \langle v, w\rangle_{Q_j} = 0\} =: H_{j,v}. \end{aligned}$$

We have the following interpretation of $H_{j,v}$ in terms of the geometry of Q_j:

Claim. *Let $v' \in \{Q_j = 0\}$ for $j = 1$ or $j = 2$. Then the image hyperplane $H_{j,v'}$ of v' under ψ_j is exactly the tangent space $T_{v'}Q_j$ to Q_j at v'.*

Proof. We recall that one characterization of $T_{v'}Q_j$ is that $w \in T_{v'}Q_j$ if and only if

$$Q_j(v' + tw) = \langle v' + tw, v' + tw\rangle_{Q_j} = \langle v', v'\rangle_{Q_j} + 2t\langle v', w\rangle_{Q_j} + t^2\langle w, w\rangle_{Q_j}$$

has no t term. Since $H_{j,v'} := \{w \in \mathbb{P}^n : \langle v', w\rangle_{Q_j} = 0\}$, the claim follows immediately. □

Now the composition $\psi_2 \circ \psi_1^{-1}$ induces an automorphism of $\mathbb{P}^{n*}$:

$$\begin{tikzcd} \mathbb{P}^{n*} \arrow[d, "\psi_1^{-1}"'] \arrow[dr, "\psi_2 \circ \psi_1^{-1}"] & \\ \mathbb{P}^n \arrow[r, "\psi_2"'] & \mathbb{P}^{n*} \end{tikzcd}$$

We use the following fact:

Claim. *Any automorphism of $\mathbb{P}^n$ (hence $\mathbb{P}^{n*}$) has a fixed point.*

Proof. Any automorphism corresponds to an element of $\mathrm{PGL}(n+1, \mathbb{C})$ and any matrix in $\mathrm{GL}(n+1, \mathbb{C})$ has at least one eigenvector. □

Hence there exists $v \in \mathbb{P}^n$ with $H_{1,v} = \psi_1(v) = \psi_2(v) = H_{2,v}$. We claim that this v is not a point of X, so that v satisfies the requirements above (with $H = H_{1,v} = H_{2,v}$). Indeed, for any $w \in X$, we have $T_w X = T_w Q_1 \cap T_w Q_2$. If we had $v \in X$, then X would be singular at v, as $T_v Q_1 = H_{1,v} = H_{2,v} = T_v Q_2$. Hence $v \notin X$, which completes the proof. $\square$

5.2 The cup product lattice on $H^{2k}(X(2)) \subset \mathbb{P}^{2k+1}$

We shall determine the middle cohomology of a smooth quadric in $\mathbb{P}^{2k+1}$, $k \geq 1$ by an iterative construction, successively taking cones over linear spaces contained in smooth quadrics of lower dimension.

Base step: Quadric in $\mathbb{P}^3$

Consider the image of the Segre embedding $s_{1,1}$ given by[1]

$$s_{1,1} : \mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow \mathbb{P}^3$$
$$([P:U],[R:S]) \mapsto [PR:US:UR:PS].$$

Naming the coordinates on the target $[X:Y:Z:W]$, any point in the image clearly lies on the quadric hypersurface $Q := \{XY - WZ = 0\}$. One checks that in fact the image is exactly the quadric Q. The Jacobian matrix of Q at point $[X:Y:Z:W]$ is $(Y, X, -Z, -W)$; this is never identically zero so Q is smooth. By lemma 5.1-a, it follows that up to a change of coordinates on $\mathbb{P}^3$ every smooth quadric $X(2)$ in $\mathbb{P}^3$ is the image of the embedding $s_{1,1}$.

By lemma 4.5, the rank of the middle cohomology $H^2(X(2))$ is 2, so it is enough to find two distinct cohomology classes in $H^2(X(2))$. Let $H_j := \left[\pi_j^{-1}(p)\right] \in H^2(\mathbb{P}^1 \times \mathbb{P}^1)$, where $j = 1, 2$, π_j are projections from $\mathbb{P}^1 \times \mathbb{P}^1$ to one of the factors and $p \in \pi_j(\mathbb{P}^1 \times \mathbb{P}^1)$ is a point in the projection. By the description of $H^\bullet(\mathbb{P}^1 \times \mathbb{P}^1, \mathbb{Z})$ in section 2.3, we have

[1] up to a permutation of coordinates on $\mathbb{P}^3$

that $\{H_j\}_{j=1,2}$ is a basis of $H^2(\mathbb{P}^1 \times \mathbb{P}^1)$. The Gram matrix of the interesection form with this choice of basis is

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Write $s := s_{1,1}$ for the Segre embedding and let $\Lambda_{1,j} := s_*(H_j) \in H^2(X(2))$, $j = 1, 2$. By the projection formula, we have

$$\Lambda_{1,1} \cdot \Lambda_{1,2} = s_*(H_1) \cdot s_*(H_2) = s_* \left(H_1 \cdot s^* \left(s_* H_2\right)\right).$$

Because s is a smooth embedding (and in particular has degree one), we have $s^* s_* H_2 = H_2$, so $\Lambda_{1,1}^2 = \Lambda_{1,2}^2 = 0$ and $\Lambda_{1,1} \cdot \Lambda_{1,2} = 1$. In particular, $\Lambda_{1,1}$ and $\Lambda_{1,2}$ are a $\mathbb{Z}$-basis for $H^2(X(2))$.

In the classical picture of the double-ruling of a smooth quadric in $\mathbb{P}^3$, the pushforwards $\Lambda_{1,j}$ of the classes H_j in $\mathbb{P}^1 \times \mathbb{P}^1$ are exactly the classes of lines contained in one of the two rulings. This description makes it visible that any lines representing the classes $\Lambda_{1,j}$ have self-intersection 0 and pairwise intersection 1. For the construction below, we write down two concrete lines in Q intersecting in a point: take the images of $\mathbb{P}^1 \times [1:0]$ and $[1:0] \times \mathbb{P}^1$ under $s_{1,1}$. The first consists of points $\{[P:0:U:0] \in Q\}$, which is the intersection of hyperplanes $\{Y = 0\}$ and $\{W = 0\}$; the second consists of points $\{[R:0:0:S] \in Q\}$, which is the intersection of hyperplanes $\{Y = 0\}$ and $\{Z = 0\}$. Call these lines $\Lambda_{1,1}$ and $\Lambda_{1,2}$, by abuse on notation ($\Lambda_{1,j}$, $j = 1, 2$ also denote the classes of the lines in $H^2(X(2))$). The intersection $\Lambda_{1,1} \cap \Lambda_{1,2}$ is the point $[1:0:0:0] \in Q$.

Cone construction

Proposition 5.1. *Let $X(2) \subset \mathbb{P}^{2k+1}$, $(k \geq 1)$ be a smooth quadric hypersurface. Then there exist k-planes $\Lambda_{k,1}$ and $\Lambda_{k,2}$ contained in $X(2)$ and intersecting along a $(k-1)$-plane Γ_k.*

Proof. Proceed by induction on k. The base case is the smooth quadric in $\mathbb{P}^3$. Suppose the proposition holds for $(k-1)$; we show that it holds for k.

Let $x \in Q$ be a point with coordinates $x = [W_0 : \cdots : W_{2k+1}]$ and let $\mathbb{T}_x Q$ be the projective tangent plane to Q at x. We recall that $\mathbb{T}_x Q$ is defined as the projective closure of a tangent plane $T_x Q$ to Q in an affine chart containing x. In the case of a hypersurface X in $\mathbb{P}^n$ defined by the homogeneous form F, $\mathbb{T}_p X$ may be seen to be cut out by the equation

$$\mathbb{T}_p X = \{[Z_0 : \cdots : Z_n] : \sum_{i=0}^{n} \frac{\partial F}{\partial Z_i}(p) Z_i = 0\}$$

(see [7] pp. 181–182). Specializing to the case of a quadric in $\mathbb{P}^{2k+1}$, we have

$$\mathbb{T}_p Q = \{[Z_0 : \cdots : Z_{2k+1}] : W_0 Z_0 + \cdots + W_{2k+1} Z_{2k+1} = 0\}.$$

Now consider $Q' := Q \cap \mathbb{T}_p Q$. Using that $\mathbb{T}_p Q \cong \mathbb{P}^{2k}$, Q' is a degree 2 hypersurface in $\mathbb{P}^{2k}$, hence a quadric. Since $\mathbb{T}_p Q' = \mathbb{T}_p Q \cap \mathbb{T}_p(\mathbb{T}_p Q) = \mathbb{T}_p Q$, Q' is clearly singular at p. Using the Jacobian criterion, it is not hard to see that p is the only singular point of Q'. By the general classification of quadrics, since the singular locus of Q' is of codimension $2k$ in $\mathbb{P}^{2k}$, it follows that Q' has rank $2k$. By lemma 5.1-b, up to a change of coordinates on $\mathbb{T}_p Q \cong \mathbb{P}^{2k}$, Q' is given by the vanishing of the form $U_0^2 + \cdots + U_{2k-1}^2$ (here $[U_0 : \cdots : U_{2k}]$ are homogeneous coordinates on $\mathbb{T}_p Q$). Hence Q' is a cone with vertex p over a smooth quadric $Q'' \subset \mathbb{P}^{2k-1} = \mathbb{P}^{2(k-1)+1}$. By induction hypothesis, there exist $(k-1)$-planes $\Lambda_{k-1,j}$, $j = 1, 2$ contained in Q'' and intersecting along a $(k-2)$-plane Γ_{k-1}. Now take $\Lambda_{k,j} := \mathrm{Cone}(\Lambda_{k-1,j}, p)$ to be the cone over $\Lambda_{k-1,j}$ with vertex p. Then $\Lambda_{k,j}$ are k-planes contained in $Q' \subset Q$ and intersecting along the $(k-1)$-plane $\mathrm{Cone}(\Gamma_{k-1}, p) =: \Gamma_k$, which completes the induction. $\square$

General case

We now show that the k-planes $\Lambda_{k,j}$, $j = 1, 2$ give a $\mathbb{Z}$-basis for $H^{2k}(X(2))$.

Theorem 5.1. *Let $Q = X(2)$ be a smooth quadric hypersurface in $\mathbb{P}^{2k+1}$ $(k \geq 1)$. Then the k-planes $\Lambda_{k,1}$, $\Lambda_{k,2}$ form a basis of $H^{2k}(Q)$, with Gram matrix of the intersection form* *equal to*

$$\begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & k \text{ even} \\ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, & k \text{ odd} \end{cases}.$$

Proof. By lemma 4.5, $H^{2k}(Q)$ has rank two. Therefore it is enough to establish the statement about the Gram matrix. For this, we perform the following two excess intersection computations:

By corollary 2.3, for all $k \geq 1$ and $j = 1, 2$

$$\begin{aligned} N_{Q/\mathbb{P}^{2k+1}} &= \mathcal{O}_{\mathbb{P}^{2k+1}}(2)\Big|_{Q}, \\ N_{\Lambda_{k,j}/\mathbb{P}^{2k+1}} &= \bigoplus_{i=1}^{k+1} \mathcal{O}_{\mathbb{P}^{2k+1}}(1)\Big|_{\Lambda_{k,j}} \quad \text{and} \\ N_{\Gamma_k/\mathbb{P}^{2k+1}} &= \bigoplus_{i=1}^{k+2} \mathcal{O}_{\mathbb{P}^{2k+1}}(1)\Big|_{\Gamma_k}, \end{aligned}$$

so that

$$\begin{aligned} c(N_{Q/\mathbb{P}^{2k+1}}) &= (1 + H|_Q), \\ c(N_{\Lambda_{k,j}/\mathbb{P}^{2k+1}}) &= (1 + H|_{\Lambda_{k,j}})^{k+1} \quad \text{and} \\ c(N_{\Gamma_k/\mathbb{P}^{2k+1}}) &= (1 + H|_{\Gamma_k})^{k+2}. \end{aligned}$$

Then by lemma 2.9, writing H for $H|_{\Gamma_k}$

$$\begin{aligned} c(N_{\Lambda_{k,j}/Q}) &= \frac{(1+H)^{k+1}}{(1+2H)} \quad \text{and} \\ c(N_{\Gamma_k/Q}) &= \frac{(1+H)^{k+2}}{(1+2H)}. \end{aligned}$$

Applying corollary 2.2, the self-intersection numbers of the k-planes $\Lambda_{k,j}$ are

$$[\Lambda_{k,j}]^2 = \int_{\Lambda_{k,j}} c(N_{\Lambda_{k,j}/Q}) = \int_{\Lambda_{k,j}} \frac{(1+H)^{k+1}}{1+2H}.$$

Expanding $1/(1+2H)$ in a power series and applying the binomial theorem (using that $H^{k+1} = 0$ in $H^\bullet(\Lambda_{k,j}, \mathbb{Z}) \cong H^\bullet(\mathbb{P}^k, \mathbb{Z})$),

$$\frac{(1+H)^{k+1}}{1+2H} = \left(\sum_{j=0}^{k} \binom{k+1}{j} H^j\right)\left(\sum_{i=0}^{k} (-2)^i H^i\right).$$

The coefficient of H^k is

$$\sum_{j=0}^{k} \binom{k+1}{k-j} (-2)^j = \frac{1}{-2} \sum_{j=0}^{k} \binom{k+1}{j+1} (-2)^{j+1} = \frac{(1-2)^{k+1}-1}{-2} = \frac{1-(-1)^{k+1}}{2} = \begin{cases} 1, & k \text{ even} \\ 0, & k \text{ odd} \end{cases}.$$

Similarly, by excess intersection formula we have (writing H for $H|_{\Gamma_k}$)

$$\begin{aligned}[\Lambda_{k,1}] \cdot [\Lambda_{k,2}] = \int_{\Gamma_k} \frac{c(N_{\Lambda_{k,1}/Q}) \cdot c(N_{\Lambda_{k,2}/Q})}{c(N_{\Gamma_k/Q})} &= \int_{\Gamma_k} \left(\frac{(1+H)^{k+1}}{1+2H}\right)^2 \left(\frac{(1+H)^{k+2}}{1+2H}\right)^{-1} \\ &= \int_{\Gamma_k} \frac{(1+H)^k}{1+2H}.\end{aligned}$$

The last term is formally identical to

$$\int_{\Lambda_{k-1,j}} \frac{(1+H)^k}{1+2H},$$

so we have

$$[\Lambda_{k,1}] \cdot [\Lambda_{k,2}] = \int_{\Gamma_k} \frac{(1+H)^k}{1+2H} = \begin{cases} 0, & k \text{ even} \\ 1, & k \text{ odd} \end{cases},$$

completing the proof. $\square$

5.3 The cup product lattice on $H^{4k}(X(2,2)) \subset \mathbb{P}^{4k+2}$

The next two sections are devoted to classifying the cup product lattices for smooth intersections of two quadrics in $\mathbb{P}^{4k+2}$ $(k \geq 1)$. To keep the notation as clear as possible, we first carry out the arguments for the case $k = 1$, $X = X(2,2) \subset \mathbb{P}^6$. Then, in the following section, we show that the arguments for the general case are simple generalizations of the arguments for case $k = 1$.

5.3.1 The case $k = 1$

Our strategy for identifying the cup product lattice on the middle cohomology $H^4(X)$ of X will be to find a set of 2-planes contained in X and take linear combinations of their Poincaré dual classes to find a generating set for $H^4(X)$. The pairwise intersections of the 2-planes computed by excess intersection formula will then determine the cup product lattice. To begin, we need to show that X contains at least one 2-plane; the remaining generators will then be constructed from this initial 2-plane.

We set up the following incidence correspondence. The space of quadrics in $\mathbb{P}^6$ is the projective space $\mathbb{P}^{27}$, where $27 = \binom{6+2}{2} - 1$—it is the projectivization of the vector space generated by monomials of degree 2 in $z_0, \ldots, z_6$ (where the z_i denote homogeneous coordinates of $\mathbb{P}^6$). Let Z be the subset of $\mathbb{G}(2,6) \times \mathbb{P}^{27}$ defined as

$$
\begin{tikzcd}
 & Z := \{(\Lambda, Q) : \Lambda \subset Q\} \subset \mathbb{G}(2,6) \times \mathbb{P}^{27} \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
\mathbb{G}(2,6) & & \mathbb{P}^{27}
\end{tikzcd}
\tag{5.1}
$$

We study the fiber $\pi_1^{-1}(\Lambda_0)$ of π_1 over a fixed 2-plane $\Lambda_0 \in \mathbb{G}(2,6)$. The restriction of π_2 to $\pi_1^{-1}(\Lambda_0)$ is clearly one-to-one, so that the fiber is isomorphic to $\pi_2(\pi_1^{-1}(\Lambda_0))$. Points of the latter correspond to quadric hypersurfaces in $\mathbb{P}^6$ containing Λ_0. Such a hypersurface is given by a homogeneous degree 2 form F in $z_0, \ldots, z_6$, with the additional requirement that the restriction $F|_{\Lambda_0}$ of F to Λ_0 is identically zero. Under the isomorphism $\Lambda_0 \cong \mathbb{P}^2$,

$F|_{\Lambda_0}$ is a homogeneous degree 2 form in w_0, w_1, w_2. We therefore have a map of vector spaces

$$T : \begin{array}{ccc} \left\{ \begin{array}{c} \text{Homogeneous degree 2 forms} \\ \text{in } z_0, \ldots, z_6 \end{array} \right\} & \longrightarrow & \left\{ \begin{array}{c} \text{Homogeneous degree 2 forms} \\ \text{in } w_0, w_1, w_2 \end{array} \right\} \\ F & \mapsto & F|_{\Lambda_0} \end{array}$$

such that the kernel of the map T may be identified with $\pi_2(\pi_1^{-1}(\Lambda_0))$. Since the map T is surjective and the image is $\binom{2+2}{2} = 6$-dimensional, the kernel is $\binom{6+2}{2} - 6 = 28 - 6 = 22$-dimensional. We conclude that the fiber $\pi_1^{-1}(\Lambda_0)$ is isomorphic to $\mathbb{P}^{21}$, hence has dimension 21.

To continue our analysis, we quote the well-known theorem on upper-semicontinuity of fibre dimension and one of its corollaries:

Theorem 5.2. *Let X be a projective variety and $\pi : X \longrightarrow \mathbb{P}^n$ a regular map; let $Y = \pi(X)$ be its image. For any $q \in Y$, let $\lambda(q) := \dim(\pi^{-1}(q))$. Then $\lambda(q)$ is a Zariski-upper-semicontinuous function of q on Y. Moreover, if $X_0 \subset X$ is any irreducible component, $Y_0 = \pi(X_0)$ and $\lambda_0 := \min_{q \in Y_0} \lambda(q)$, then*

$$\dim(X_0) = \dim(Y_0) + \lambda_0.$$

Proof. [7] pp. 139–141. The statement of the theorem is Corollary 11.13 on page 138. □

Corollary 5.1. *Let $\pi : X \longrightarrow Y$ be a regular map of projective varieties, with Y irreducible. If all fibres $\pi^{-1}(p)$ are irreducible of the same dimension, then X is irreducible.*

Proof. [7] p. 139. The statement is Theorem 11.14 on page 138. □

Before we can use the above theorem to further analyze the correspondence (5.1), we need to demonstrate that Z is Zariski-closed in $\mathbb{G}(2,6) \times \mathbb{P}^{27}$. Towards this end, let

$$\Gamma_1 := \{(\Lambda, p) : p \in \Lambda\} \subset \mathbb{G}(2,6) \times \mathbb{P}^6 \text{ and}$$
$$\Gamma_2 := \{(Q, p) : p \in Q\} \subset \mathbb{P}^{27} \times \mathbb{P}^6$$

be the universal families over $\mathbb{G}(2,6)$ and $\mathbb{P}^{27}$, respectively (in particular, Γ_1 and Γ_2 are closed subsets of $\mathbb{G}(2,6) \times \mathbb{P}^6$ and $\mathbb{P}^{27} \times \mathbb{P}^6$, respectively). Let Γ be defined as

$$\begin{tikzcd}[column sep=small] & \Gamma := \{(\Lambda, Q, p) : p \in \Lambda \cap Q\} \subset \mathbb{G}(2,6) \times \mathbb{P}^{27} \times \mathbb{P}^6 \arrow[dl, "\pi_{1,3}"'] \arrow[dr, "\pi_{2,3}"] & \\ \mathbb{G}(2,6) \times \mathbb{P}^6 & & \mathbb{P}^{27} \times \mathbb{P}^6 \end{tikzcd},$$

where $\pi_{1,3}$ and $\pi_{2,3}$ are projection maps. Then as a set Γ is equal to $\pi_{1,3}^{-1}(\Gamma_1) \cap \pi_{2,3}^{-1}(\Gamma_2)$, hence is a closed subset of $\mathbb{G}(2,6) \times \mathbb{P}^{27} \times \mathbb{P}^6$. Now let $\pi := \pi_{1,2}|_\Gamma : \Gamma \longrightarrow \mathbb{G}(2,6) \times \mathbb{P}^{27}$ be the restriction of the projection $\pi_{1,2}$ to Γ. For any $\Lambda \in \mathbb{G}(2,6)$ and $Q \in \mathbb{P}^{27}$, we have $\pi^{-1}(\Lambda, Q) = \Lambda \cap Q$, hence $\dim(\pi^{-1}(\Lambda, Q)) \leq 2$. Then by theorem 5.2

$$Z = \{(\Lambda, Q) \in \mathbb{G}(2,6) \times \mathbb{P}^{27} : \dim(\pi^{-1}(\Lambda, Q)) = 2\} = \{(\Lambda, Q) \in \mathbb{G}(2,6) \times \mathbb{P}^{27} : \dim(\pi^{-1}(\Lambda, Q)) \geq 2\}$$

is a Zariski-closed subset of $\mathbb{G}(2,6) \times \mathbb{P}^{27}$, as desired.

We can now apply theorem 5.2 to the correspondence (5.1). As $\mathbb{G}(2,6)$ is irreducible and all fibres of π_1 are isomorphic to $\mathbb{P}^{21}$ (hence irreducible of the same dimension) it follows by corollary 5.1 that Z is irreducible. Then Z is $\dim \mathbb{G}(2,6) + 21 = 2(6-2) + 21 = 33$-dimensional by the second part of theorem 5.2.

Now, it is well-known that a quadric hypersurface of dimension n contains linear subspaces of each dimension less than or equal to the integer part of $n/2$, so π_2 is surjective. It follows from the second part of theorem 5.2 that for any $Q \in \mathbb{P}^{27}$, the fibre $\pi_2^{-1}(Q)$ has dimension $\geq 33 - 27 = 6$. We'd like to show that the generic fibre is six-dimensional. Let $V_n := \{Q \in \mathbb{P}^{27} : \dim \pi_2^{-1}(Q) \geq n\}$; this is a Zariski-closed subset of $\mathbb{P}^{27}$ by upper-semicontinuity of fibre dimension. Moreover by above, $V_m = \mathbb{P}^{27}$ for all $m \leq 6$. Then

$U := \{Q \in \mathbb{P}^{27} : \dim \pi_2^{-1}(Q) = 6\} = V_6 \setminus V_7 = \mathbb{P}^{27} \setminus V_7$ (the latter set is nonempty as by construction the minimum fibre dimension 6 is reached over at least one $Q \in \mathbb{P}^{27}$). Since V_7 is a Zariski-closed subset of $\mathbb{P}^{27}$, this shows that U is a nonempty open dense subset of $\mathbb{P}^{27}$. In this sense the generic fibre is six-dimensional.

For any $Q \in \mathbb{P}^{27}$ corresponding to a quadric in $\mathbb{P}^6$, we set the notation $\Gamma_Q := \{\Lambda \in \mathbb{G}(2,6) : \Lambda \subset Q\} = \pi_1(\pi_2^{-1}(Q))$. The restriction of π_1 to $\pi_2^{-1}(Q)$ is one-to-one, so that Γ_Q is again 6-dimensional for each $Q \in U$. As $\dim \mathbb{G}(2,6) = 12$, we conclude that for any quadrics Q, Q' in U, Γ_Q and $\Gamma_{Q'}$ have complementary dimension in $\mathbb{G}(2,6)$. Next, we show that this statement continues to hold for all *smooth* quadrics in $\mathbb{P}^6$. Towards this end, we have the following lemma

Lemma 5.2. *For any smooth quadric hypersurfaces $Q, Q' \subset \mathbb{P}^6$, we have $[\,\Gamma_Q\,] = [\,\Gamma_{Q'}\,]$ in $H^6(\mathbb{G}(2,6),\mathbb{Z})$.*

Proof. By lemma 5.1-b, there exist $g, h \in \mathrm{PGL}(n+1,\mathbb{C})$ such that under the change of coordinates of $\mathbb{P}^n$ induced by g and h, Q and Q' respectively are defined by the vanishing of the form $\sum_{i=0}^{6} Z_i$. Since $\mathrm{PGL}(n+1,\mathbb{C})$ is path-connected, there exist continuous paths γ from $\mathrm{id} \in \mathrm{PGL}(n+1,\mathbb{C})$ to g and $\tilde{\gamma}$ from h to id, respectively. Then composition of the paths γ and $\tilde{\gamma}$ induces a continuous deformation of Q to Q'. Finally, as elements of $\mathrm{PGL}(n+1,\mathbb{C})$ take k-planes to k-planes, restricting this deformation to 2-planes contained in Q induces a continuous deformation of Γ_Q to $\Gamma_{Q'}$. Since cohomology is a discrete invariant, it remains constant under continuous deformations and the lemma follows. $\square$

Now, it is well-known that the set of singular quadrics in $\mathbb{P}^6$ is parametrized by a hypersurface in $\mathbb{P}^{27}$. In particular, the set of smooth quadrics in $\mathbb{P}^6$ is parametrized by an open subset of $\mathbb{P}^{27}$. Since any two nonempty open subsets of $\mathbb{P}^{27}$ have nonempty intersection, we conclude that there exists a smooth quadric $Q_0 \in U \subset \mathbb{P}^{27}$, where U is the open set of quadrics $Q \in \mathbb{P}^{27}$ for which the fibre $\pi_2^{-1}(Q)$ is 6-dimensional. Then Γ_Q is a six-dimensional subset of $\mathbb{G}(2,6)$, so that $[\Gamma_{Q_0}] \in H^{12}(\mathbb{G}(2,6),\mathbb{Z})$. But by corollary 5.2, if Q is any smooth quadric, we have $[\Gamma_Q] = [\Gamma_{Q_0}]$. In particular, Γ_Q is six-dimenisonal for any smooth quadric Q. Hence any intersection of two smooth quadrics in $\mathbb{P}^6$ contains finitely many 2-planes. Our next goal is to show that this number is not zero.

Lemma 5.3. *The intersection of two smooth quadrics in $\mathbb{P}^6$ contains at least one 2-plane.*

Proof. We put our knowledge of the additive structure of $H^\bullet(\mathbb{G}(2,6),\mathbb{Z})$ and the identity (2.9) to work. There are five sequences a_0, a_1, a_2 satisfying $0 \le a_0 \le a_1 \le a_2 \le 4$ and $a_0 + a_1 + a_2 = 6$. These are

a_0, a_1, a_2	$4-a_2, 4-a_1, 4-a_0$
$0,2,4$	$0,2,4$
$0,3,3$	$1,1,4$
$1,1,4$	$0,3,3$
$1,2,3$	$1,2,3$
$2,2,2$	$2,2,2$

.

Hence

$$[\,\Gamma_Q\,] = [\,\Gamma_{Q'}\,] = a\,\sigma_{0,2,4} + b\,\sigma_{0,3,3} + c\,\sigma_{1,1,4} + d\,\sigma_{1,2,3} + e\,\sigma_{2,2,2} \quad \text{for some } a,b,c,d,e \in \mathbb{Z}_{\ge 0}$$

and

$$[\,\Gamma_Q\,]\cdot[\,\Gamma_{Q'}\,] = (a^2 + 2bc + d^2 + e^2)[\,\mathrm{pt.}\,],$$

where $[\,\mathrm{pt.}\,]$ denotes the class of a point in $H^{12}(\mathbb{G}(2,6),\mathbb{Z})$.

To show existence of a 2-plane in $Q \cap Q'$, it is enough to show that $(a^2+2bc+d^2+e^2) > 0$, as this will imply that Γ_Q and $\Gamma_{Q'}$ have nonempty intersection. In particular, showing that $d > 0$ will suffice. To find d, we use the identity (2.9) once again: we have

$$d = (a\,\sigma_{0,2,4} + b\,\sigma_{0,3,3} + c\,\sigma_{1,1,4} + d\,\sigma_{1,2,3} + e\,\sigma_{2,2,2})\cdot\sigma_{1,2,3} = [\,\Gamma_Q\,]\cdot\sigma_{1,2,3}.$$

We may choose a complete projective flag $\mathcal{F} = \mathbb{P}^0 \subset \cdots \subset \mathbb{P}^6$ such that $\mathbb{P}^1 \cap Q$ consists of two points, and $\mathbb{P}^3 \cap Q$ and $\mathbb{P}^5 \cap Q$ are smooth quadrics in $\mathbb{P}^3$ and $\mathbb{P}^5$, respectively. The conditions imposed on a plane Λ to lie in the intersection of Γ_Q and $\overline{S^{\mathcal{F}}_{1,2,3}}$ are:

1. $\Lambda \subset Q$
2. Λ intersects $\mathbb{P}^1$ in a point
3. Λ intersects $\mathbb{P}^3$ in a line
4. Λ is contained in $\mathbb{P}^5$.

We claim that there are a total of 8 such 2-planes Λ.

First, $\mathbb{P}^1 \cap Q$ consists of two points. By conditions (1) and (2), Λ must contain one of these points. Choose one of the points, say p.

Now, $\mathbb{P}^3 \cap Q$ is a smooth quadric in $\mathbb{P}^3$, which has a double-ruling seen in the previous section. Therefore there are two lines contained in $\mathbb{P}^3 \cap Q$ passing through p. By conditions (1), (2) and (3), Λ must contain one of these lines. Choose one of the lines, say ℓ.

Let V and W be the 6- and 2-dimensional vector spaces corresponding to the $\mathbb{P}^5$ part of the flag $\mathcal{F}$ and the line ℓ, respectively. Let F be the symmetric bilinear form associated to the quadric $Q' := \mathbb{P}(V) \cap Q$. By our choice of flag, F is nondegenerate as Q' is smooth. Because $\ell \subset \mathbb{P}(V) \cap Q$, we have $F|_W \equiv 0$.

Each 2-plane Λ containing the line ℓ and contained in Q' corresponds to a 3-dimensional vector subspace P of V containing W such that $F|_P \equiv 0$. Now 3-dimensional subspaces of V containing W are in bijection with lines in V/W. As F is nondegenerate, $W^\perp$ is a $6 - 2 = 4$-dimensional subspace of V containing W. Hence, $W^\perp/W$ is a 2-dimensional subspace of V/W. It is not hard to see that F descends to a well-defined nondegenerate bilinear form on $W^\perp/W$. Projectivizing, F defines a quadric in $\mathbb{P}(W^\perp/W) \cong \mathbb{P}^1$, that is, two points. Choosing either point gives a line in $W^\perp/W$ corresponding to a 2-plane Λ satisfying conditions (1)–(4) above.

Now suppose toward contradiction that U is a choice of 3-dimensional subspace of V containing W, with $F|_U \equiv 0$ and with $U \cap W^\perp \neq U$. Let $p \in U \setminus W^\perp$ and let $w \in W$ be such that $F(p,w) \neq 0$. Then $F(p+w,p+w) = F(p,p) + F(w,w) + 2F(p,w) = 2F(p,w) \neq 0$. Since $p + w \in U$, this is the desired contradiction.

Therefore the number of 2-planes satisfying (1)–(4) is $2^3 = 8$. $\square$

Construction of generators of the cup product lattice on $H^4(X(2,2))$, I

Let $X = X(2,2)$ be a smooth complete intersection of two quadrics in $\mathbb{P}^6$. By a Bertini argument, we may assume that the two defining quadrics are smooth, so that by lemma 5.3 X contains a 2-plane, call it P. Starting with P, we now construct seven 2-planes $\Lambda_0, \ldots, \Lambda_6$ contained in X, each intersecting P in codimension one and pairwise intersecting in codimension 2. Using the excess intersection formula to compute the pairwise cup products, we then show that certain linear combinations of the classes of the eight 2-planes $[P], [\Lambda_0], \ldots, [\Lambda_6]$ along with $[H]^2$, the square of the class of a hyperplane section of X, form a set of generators for $H^4(X)$.

By lemma 5.1-d, up to a change of coordinates on $\mathbb{P}^6$, X is given as the intersection of hypersurfaces defined by the forms

$$Q = \sum_{i=0}^{6} Z_i \quad \text{and} \quad Q' = \sum_{i=0}^{6} a_i Z_i \quad a_i \neq 0 \text{ for all } i, \quad a_i \neq a_j \text{ for } i \neq j$$

in $\mathbb{P}^6$. For $i = 0, \ldots, 6$, let π_i be the rational map

$$\mathbb{P}^6 \smallsetminus [0 : \cdots : 1 : \cdots 0] \longrightarrow \mathbb{P}^5$$

given by projection to the hyperplane $\{Z_i = 0\}$ (equivalently, given by omitting the i-th coordinate). None of the points $[0 : \cdots : 1 : \cdots : 0]$ (here the i-th coordinate is 1 and the rest 0) are on X (this is easily seen, say, from the non-vanishing of the form Q at these points), so each π_i restricts to a regular map on X. Let $W_0 = Z_0, \ldots, W_{i-1} = Z_{i-1}, W_i = Z_{i+1}, \ldots, W_5 = Z_6$ be the homogeneous coordinates induced on $\mathbb{P}^5$ by identifying it with the hyperplane $\{Z_i = 0\}$ in X.

Claim. *The restriction $\pi_i|_X : X \longrightarrow \mathbb{P}^5$ is a $2:1$ map ramified over the intersection of X with the hyperplane $\{Z_i = 0\}$, with image a smooth quadric in $\mathbb{P}^5$*

Proof. To be in the image of $\pi_i|_X$, $[W_0 : \cdots : W_5]$ must have the property that there exists a complex number Z_i such that

$$W_0^2 + \cdots + W_5^2 + Z_i^2 = 0 \quad \text{and} \quad a_0 W_0^2 + \cdots + a_{i-1} W_{i-1}^2 + a_i Z_i^2 + a_{i+1} W_i^2 + \cdots + a_6 W_5^2 = 0.$$

Solving for Z_i^2, we find the simultaneous conditions

$$Z_i^2 = -\left(W_0^2 + \cdots + W_5^2\right) \quad \text{and} \quad Z_i^2 = -\frac{1}{a_i}\left(a_0 W_0^2 + \cdots + a_{i-1} W_{i-1}^2 + a_{i+1} W_i^2 + \cdots + a_6 W_5^2\right).$$

A common solution exists if and only if

$$a_i(W_0^2 + \cdots + W_5^2) = a_0 W_0^2 + \cdots + a_{i-1} W_{i-1}^2 + a_{i+1} W_i^2 + \cdots + a_6 W_5^2,$$

which is a quadric in $\mathbb{P}^5$, defined as the vanishing locus of the form

$$F := (a_0 - a_i) W_0^2 + \cdots + (a_{i-1} - a_i) W_{i-1}^2 + (a_{i+1} - a_i) W_i^2 + \cdots + (a_6 - a_i) W_5^2.$$

Taking partials, we find

$$\frac{\partial F}{\partial W_j} = \begin{cases} 2(a_j - a_i) W_j\,, & j < i \\ 2(a_{j+1} - a_i) W_j\,, & j \geq i \end{cases}.$$

Because $a_i \neq a_j$ for $i \neq j$ by the initial choice of coordinates, it follows by the Jacobian criterion that the singular locus of F is the intersection of the coordinate hyperplanes $\{W_j = 0\},\ j = 0, \ldots, 5$ in $\mathbb{P}^5$, hence empty. So the quadric $\{F = 0\} \subset \mathbb{P}^5$ is smooth.

Now, let $p := [W_0 : \cdots : W_5]$ be a point of $\{F = 0\}$. Look at $\pi_i|_X^{-1}(p)$. The coordinates of each point $[W_0 : \cdots : Z_i : \cdots : W_6] \in \pi_i|_X^{-1}(p) \subset X$ must satisfy $Z_i^2 = -(W_0^2 + \cdots + W_5^2) =: \beta$. The last equation has two distinct solutions for Z_i if β is nonzero and a unique solution for Z_i if β is zero. It follows that the map $\pi_i|_X$ is $2:1$, ramified over $\{Z_i = 0\} \cap X$. $\square$

Now look at the restriction of the projection $\pi_i|_X$ to the 2-plane $P \subset X$. We show that this map is an inclusion. Since $\pi_i|_X$ has fibers consisting of one or two points over each $p \in \pi_i|_X(X)$ by the above claim, we have that the degree $\deg(\pi_i|_P)$ of the restriction of π_i to P is either 1 or 2. The image $\pi_i|_P$ is a linear space and in fact of dimension 2, again because of the fact that $\pi_i|_X$ has everywhere-finite fibers. Using the isomorphism $P \cong \mathbb{P}^2$, $\pi_i|_P : \mathbb{P}^2 \to \mathbb{P}^5$ is given by homogeneous forms $F_0, \ldots, F_5$, each of some fixed degree d. We have the equality

$$\deg(\pi_i|_P)\deg(\pi_i|_P(P)) = d^2,$$

which may be seen as follows: $\deg(\pi_i|_P(P))$ is by definition the number of points of intersection of $\pi_i|_P(P)$ with a generic codimension 2 linear subspace of $\mathbb{P}^5$. We may realize a codimension 2 linear subspace as the intersection of two hyperplanes H_1, $H_2 \subset \mathbb{P}^5$. The pullback of H_j to $\mathbb{P}^2$ under $\pi_i|_P$ is a degree d hypersurface and by Bezout the intersection $\pi_i|_P(H_1) \cap \pi_i|_P(H_2)$ consists of d^2 points. Now the map $\pi_i|_P$ is generically $\deg(\pi_i|_P)$-to-one, so that the number of points of intersection of the image of $\mathbb{P}^2$ under $\pi_i|_P$ with a generic codimension 2 linear subspace is $d^2/\deg(\pi_i|_P)$, as desired. Since the image of P is a linear space, so we have $\deg(\pi_i|_P(P)) = 1$, giving the necessary condition $\deg(\pi_i|_P) = d^2$. The equation $2 = d^2$ has no solutions in integers, so necessarily $\deg(\pi_i|_P) = 1$ and so $d = 1$. It follows that $\pi_i|_P$ is an inclusion.

Finally, consider $Y_i := \pi_i|_X^{-1}(\pi_i|_X(P)) \subset X$. Being the inverse image under a $2:1$ map of a degree 1 variety, Y_i has degree 2; being the inverse image of a pure-2-dimensional variety under a finite map, Y_i has pure dimension 2. Moreover, Y_i contains the 2-plane P as an irreducible component. By degree considerations, Y_i contains exactly one irreducible component other than P, necessarily of degree 1. Moreover, this other component must intersect P along the section of P by the ramification locus of $\pi_i|_X$, that is, intersecting P along the codimension 1 linear subpace $\{Z_i = 0\} \cap P$. Because the second component has degree 1, it is necessarily a linear space (of dimension 2). This is the desired new

2-plane: call it Λ_i.

We have constructed 2-planes $\Lambda_0, \ldots, \Lambda_6$ intersecting P along distinct codimension 1 linear subspaces, hence pairwise distinct and distinct from P. It will be necessary to know the codimension of pairwise intersections $\Lambda_i \cap \Lambda_j$, $i \neq j$ in Λ_i and Λ_j:

Claim. *For any $i \neq j$, Λ_i and Λ_j intersect along a codimension 2 linear subspace.*

Proof. By above, P intersects Λ_i along the codimension 1 linear subspace $P \cap \{Z_i = 0\} = \Lambda_i \cap \{Z_i = 0\}$. Therefore, the codimension 2 linear subspace

$$\Lambda_i \cap \{Z_i = 0\} \cap \{Z_j = 0\} = P \cap \{Z_i = 0\} \cap \{Z_j = 0\} = \Lambda_j \cap \{Z_i = 0\} \cap \{Z_j = 0\}$$

is contained in $\Lambda_i \cap \Lambda_j$.

Suppose toward contradiction that Λ_i and Λ_j intersect along a codimension 1 linear subspace, write $\Lambda_i \cap \Lambda_j =: \tilde{\Lambda}$. We have that $\tilde{\Lambda} \cap P = P \cap \{Z_i = 0\} \cap \{Z_j = 0\}$ and $\tilde{\Lambda} \cap P$ is a codimension 1 linear subspace of $\tilde{\Lambda}$. Let L be the linear subspace spanned by $\tilde{\Lambda}$ and P—this is a 3-plane containing Λ_1, Λ_2 and P as codimension 1 linear subspaces.

Now consider $Y := L \cap \{Q = 0\}$, where Q is one of the quadratic forms defining X. If L is not contained in the locus $\{Q = 0\}$, then Y is a dimension 2 (not necessarily smooth) quadric hypersurface of L containing three 2-planes Λ_1, Λ_2 and P, which is impossible. So $L \subseteq \{Q = 0\}$. But $\{Q = 0\}$ is a smooth dimension 5 quadric hypersurface in $\mathbb{P}^6$, hence can't contain a 3-dimensional linear subspace. This is the desired contradiction. $\square$

Proposition 5.2. *Let $X = X(2,2) \subset \mathbb{P}^{2k+2}$ be a smooth complete intersection of type (2,2) and dimension $2k$. Let Λ_1 and Λ_2 be k-planes contained in X and write $\Gamma = \Lambda_1 \cap \Lambda_2$ for the linear space formed by their intersection. Suppose that Γ is ℓ-dimensional. Then*

$$[\Lambda_1] \cdot [\Lambda_2] = (-1)^{\ell} \left\lfloor 1 + \frac{\ell}{2} \right\rfloor,$$

where $\lfloor \cdot \rfloor$ denotes the integer part of $\cdot$.

Proof. By corollary 2.3, we have

$$\begin{aligned}
N_{X/\mathbb{P}^{2k+2}} &= \left(\mathcal{O}_{\mathbb{P}^{2k+2}}(2) \oplus \mathcal{O}_{\mathbb{P}^{2k+2}}(2)\right)\Big|_X, \\
N_{\Lambda_i/\mathbb{P}^{2k+2}} &= \left(\bigoplus_{i=1}^{k+2} \mathcal{O}_{\mathbb{P}^{2k+2}}(1)\right)\Bigg|_{\Lambda_i}, \quad i = 1, 2 \quad \text{and} \\
N_{\Gamma/\mathbb{P}^{2k+2}} &= \left(\bigoplus_{i=1}^{2k+2-\ell} \mathcal{O}_{\mathbb{P}^{2k+2}}(1)\right)\Bigg|_{\Gamma}.
\end{aligned}$$

For a variety $X \subset \mathbb{P}^n$, let H_X denote the section $X \cap \mathbb{P}^{n-1}$ of X by a hyperplane.

The total Chern classes of the normal bundles above are

$$\begin{aligned}
c(N_{X/\mathbb{P}^{2k+2}}) &= (1 + 2H_X)^2, \\
c(N_{\Lambda_i/\mathbb{P}^{2k+2}}) &= (1 + H_{\Lambda_i})^{k+2}, \quad i = 1, 2 \\
c(N_{\Gamma/\mathbb{P}^{2k+2}}) &= (1 + H_\Gamma)^{2k+2-\ell}.
\end{aligned}$$

Applying lemma 2.9, we have

$$\begin{aligned}
c(N_{\Lambda_i/X}) &= \frac{(1 + H_{\Lambda_i})^{k+2}}{(1 + 2H_{\Lambda_i})^2}, \quad i = 1, 2 \quad \text{and} \\
c(N_{\Gamma/X}) &= \frac{(1 + H_\Gamma)^{2k+2-\ell}}{(1 + 2H_\Gamma)^2}.
\end{aligned}$$

By excess intersection formula, we then have (writing H for H_Γ)

$$\begin{aligned}
[\Lambda_1] \cdot [\Lambda_2] &= \int_\Gamma \left(\frac{(1+H)^{k+2}}{(1+2H)^2}\right)^2 \left(\frac{(1+H)^{2k+2-\ell}}{(1+2H)^2}\right)^{-1} \\
&= \int_\Gamma \frac{(1+H)^{\ell+2}}{(1+2H)^2}.
\end{aligned}$$

Therefore, we are after the coefficient of H^ℓ in

$$\frac{(1+H)^{\ell+2}}{(1+2H)^2}.$$

Dividing $(1+H)^{\ell+2}$ by $(1+2H)^2$ with remainder, we have

$$(1+H)^{\ell+2} = p(H)(1+2H)^2 + q(H), \quad \deg p = \ell,\ \deg q \leq 1. \tag{5.2}$$

Expanding q in a Taylor series about $-1/2$, we have

$$q(H) = q\left(-\frac{1}{2}\right) + q'\left(-\frac{1}{2}\right)\left(H + \frac{1}{2}\right).$$

Letting $H = -1/2$ in (5.2), we find

$$\left(\frac{1}{2}\right)^{\ell+2} = 0 + q\left(-\frac{1}{2}\right),$$

hence $q(-1/2) = 1/2^{\ell+2}$. Differentiating (5.2) with respect to H, find

$$(\ell+2)(1+H)^{\ell+1} = p'(H)(1+2H)^2 + 4p(H)(1+2H) + q'(H). \tag{5.3}$$

Letting $H = -1/2$ in (5.3),

$$(\ell+2)\left(\frac{1}{2}\right)^{\ell+1} = 0 + 0 + q'\left(-\frac{1}{2}\right).$$

So that, putting the above together, we have

$$q(H) = \frac{1}{2^{\ell+2}} + \frac{\ell+2}{2^{\ell+1}}\left(H + \frac{1}{2}\right) = \frac{\ell+2}{2^{\ell+1}}H + \frac{\ell+3}{2^{\ell+2}}.$$

We now determine the coefficient of H^ℓ in $p(H) = p_\ell H^\ell + \cdots + p_1 H + p_0$. Since p_ℓ is also the leading coefficient of $p(H)$, we may recover it from the leading coefficient of $p(H)(1+2H)^2$ by comparing the latter with the leading coefficient of $(1+H)^{\ell+2}$ and using the equality (5.2) (because $\ell+2 \geq 2$, the terms in q don't contribute to the coefficient of $H^{\ell+2}$ in the right side of (5.2)). Clearly the coefficient of $H^{\ell+2}$ of $(1+H)^{\ell+2}$ is 1. Expanding the square in $(1+2H)^2$, we have $p(H)(1+2H)^2 = 4p_\ell H^{\ell+2} + \text{other terms}$. Comparing coefficients, we have $p_\ell = 1/4$.

Now divide both sides of (5.2) by $(1+2H)^2$:

$$\frac{(1+H)^{\ell+2}}{(1+2H)^2} = p(H) + \frac{q(H)}{(1+2H)^2}.$$

Expanding $1/(1+2H)^2$ in a series,

$$\frac{q(H)}{(1+2H)^2} = \left(\frac{\ell+2}{2^{\ell+1}}\,H + \frac{\ell+3}{2^{\ell+2}}\right)\left(\sum_{j=0}^{\ell}(j+1)(-2)^j H^j\right),$$

where the series expansion is truncated at the ℓ-th term because $H^j = 0$ for all $j > \ell$ in $H^\bullet(\Gamma,\mathbb{Z})$. We then have

$$\frac{q(H)}{(1+2H)^2} = \left\{\frac{\ell+2}{2^{\ell+1}}\left(\ell(-1)^{\ell-1}2^{\ell-1}\right) + \frac{\ell+3}{2^{\ell+2}}\left((\ell+1)(-1)^\ell 2^\ell\right)\right\} H^\ell + \text{other terms.}$$

Simplifying the coefficient, we have

$$\begin{aligned}\frac{\ell+2}{2^{\ell+1}}\left(\ell(-1)^{\ell-1}2^{\ell-1}\right) + \frac{\ell+3}{2^{\ell+2}}\left((\ell+1)(-1)^\ell 2^\ell\right) &= (-1)^\ell\left(\frac{(\ell+1)(\ell+3)}{4} - \frac{\ell(\ell+2)}{4}\right)\\ &= (-1)^\ell\,\frac{2\ell+3}{4}.\end{aligned}$$

Putting everything together,

$$\frac{(1+H)^{\ell+2}}{(1+2H)^2} = \left(\frac{1}{4} + (-1)^\ell\,\frac{2\ell+3}{4}\right)H^\ell + \text{other terms} = (-1)^\ell\left(\frac{(-1)^\ell+3}{4} + \frac{\ell}{2}\right)H^\ell + \text{other terms.}$$

We are done if we can show that

$$\left(\frac{(-1)^\ell+3}{4} + \frac{\ell}{2}\right) = \left\lfloor 1 + \frac{\ell}{2}\right\rfloor \quad \text{for all } \ell \geq 0.$$

But this is easily checked by considering the cases ℓ even and ℓ odd separately. $\square$

Construction of generators of the cup product lattice on $H^4(X(2,2))$, II

We now apply proposition 5.2 to find the pairwise cup products of the 2-planes $P, \Lambda_0, \ldots, \Lambda_6$ contained in a smooth complete intersection of type $(2,2)$ in $\mathbb{P}^6$ constructed previously. We have

$$[P] \cdot [P] = (-1)^2 \left\lfloor 1 + \frac{2}{2} \right\rfloor = 2,$$
$$[\Lambda_i] \cdot [\Lambda_i] = (-1)^2 \left\lfloor 1 + \frac{2}{2} \right\rfloor = 2, \qquad i = 0, \ldots, 6$$

$$[P] \cdot [\Lambda_i] = (-1)^1 \left\lfloor 1 + \frac{1}{2} \right\rfloor = -1, \qquad i = 0, \ldots, 6$$
$$[\Lambda_i] \cdot [\Lambda_j] = (-1)^0 \left\lfloor 1 + \frac{0}{2} \right\rfloor = 1, \qquad i, j = 0, \ldots, 6, \quad i \neq j.$$

Moreover, if $H \subset X$ is a hyperplane section of X, we have

$$[P] \cdot [H]^2 = \deg(P) = 1,$$
$$[\Lambda_i] \cdot [H]^2 = \deg(\Lambda_i) = 1, \quad i = 0, \ldots, 6$$
$$[H]^2 \cdot [H]^2 = \deg(X) = \deg(Q_1) \deg(Q_2) = 4.$$

The cup product lattice on $H^4(X(2,2))$

We now have enough information to prove the following:

Theorem 5.3. *The cup product lattice $H^4(X)$ of a smooth complete intersection X of type (2,2) in $\mathbb{P}^6$ is isomorphic to the Γ_8 $(= E_8)$ lattice.*

For clarity, in the proof below we denote the intersection form on $H^4(X)$ by $(\alpha, \beta) \mapsto \alpha \cup \beta$ instead of $(\alpha, \beta) \mapsto \alpha \cdot \beta$ to distinguish it from the bilinear form of the Γ_8 lattice.

Proof. We recall that in lemma 2.3 we wrote down generators $\gamma_1, \ldots, \gamma_9$ for Γ_8. The idea of the proof is to identify the elements γ_i with the following classes g_i in $H^4(X)$:

$g_i := ([\Lambda_{i-1}] - [\Lambda_i])$ for $i = 1, \ldots, 6$, $g_7 := \left([P] - [H]^2 + [\Lambda_6]\right)$, $g_8 := [H]^2$ and $g_9 := [P]$.

Suppose for the moment that we've shown the following claim.

Claim (*). *We have* $\gamma_i \cdot \gamma_j = g_i \cup g_j$ *for all pairs* i, j.

Let $h : \Gamma_8 \to H^4(X)$ be the map sending $\sum a_i \gamma_i$ to $\sum a_i g_i$. Since $\{\gamma_i\}$ is only a generating set, there may exist $v \in \Gamma_8$ that can be written as a linear combination of γ_i in two different ways: $v = \sum a_i \gamma_i = \sum b_i \gamma_i$ with $a_i \neq b_i$ for some i. We verify that in this case also $\sum a_i g_i = \sum b_i g_i$ in $H^4(X)$, so that h is well-defined. Indeed, we have $0 = \sum (a_i - b_i)\gamma_i$. Then by bilinearity and claim (*),

$$\begin{aligned} 0 &= \left(\sum (a_i - b_i)\gamma_i\right) \cdot \left(\sum (a_i - b_i)\gamma_i\right) \\ &= \left(\sum (a_i - b_i) g_i\right) \cup \left(\sum (a_i - b_i) g_i\right). \end{aligned}$$

But $H^4(X)$ is positive-definite, so that $\sum_i (a_i - b_i) g_i = 0$. So h is well-defined.

Proof of claim (*). Let $G := (\gamma_i \cdot \gamma_j)_{i,j=1}^9$, where $\cdot$ is the form on Γ_8. We have

$$G = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2 & 4 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 \end{pmatrix}.$$

By symmetry of $\cup$, it is enough to check equality of the terms above and including the diagonal. Let $1 \le i < j \le 6$. We have

$$g_i \cup g_j = [\Lambda_{i-1}] \cup [\Lambda_{j-1}] - [\Lambda_{i-1}] \cup [\Lambda_j] - [\Lambda_i] \cup [\Lambda_{j-1}] + [\Lambda_i] \cup [\Lambda_j] = \begin{cases} 2-1-1+2=2 & i=j \\ 1-1+2-1=-1 & i=j-1 \\ 1-1-1+1=0 & i<j-1 \end{cases}.$$

The remaining three cases with the first term equal to g_i, $1 \le i \le 6$ are

$$\begin{aligned} g_i \cup g_7 &= ([\Lambda_{i-1}] - [\Lambda_i]) \cup \left([P] - [H]^2 + [\Lambda_6]\right) \\ &= \begin{cases} 0 \text{ (by symmetry)}, & 1 \le i \le 5 \\ ([\Lambda_5] - [\Lambda_6]) \cup [\Lambda_6] = 1 - 2 = -1, & i = 6 \end{cases}, \\ g_i \cup g_8 &= ([\Lambda_{i-1}] - [\Lambda_i]) \cup [H]^2 = 0 \text{ (by symmetry)}, \\ g_i \cup g_9 &= ([\Lambda_{i-1}] - [\Lambda_i]) \cup [P] = 0 \text{ (by symmetry)}. \end{aligned}$$

The remaining g_7 cup products are

$$\begin{aligned} g_7 \cup g_7 &= \left([P] - [H]^2 + [\Lambda_6]\right) \cup \left([P] - [H]^2 + [\Lambda_6]\right) = (2-1-1) - (1-4+1) + (-1-1+2) = 2, \\ g_7 \cup g_8 &= \left([P] - [H]^2 + [\Lambda_6]\right) \cup [H]^2 = 1 - 4 + 1 = -2, \\ g_7 \cup g_9 &= \left([P] - [H]^2 + [\Lambda_6]\right) \cup [P] = 2 - 1 - 1 = 0. \end{aligned}$$

Finally,

$$\begin{aligned} g_8 \cup g_8 &= [H]^2 \cup [H]^2 = 4, \\ g_8 \cup g_9 &= [H]^2 \cup [P] = 1, \\ g_9 \cup g_9 &= [P] \cup [P] = 2. \end{aligned}$$

Therefore also $G = (g_i \cup g_j)_{i,j=1}^9$, which completes the proof of claim (*). $\square$

By bilinearity of $\cdot$ and $\cup$ and claim (*), h is in fact a map of lattices, i.e. $v \cdot w =$

$h(v) \cup h(w)$ for any $v, w \in \Gamma_8$. If $w \in \ker(h)$, then $w \cdot w = 0$, so $h(w) \cup h(w) = 0$ and $h(w) = 0$, hence $\ker(h) = 0$. (here we again use that the lattices Γ_8 and $H^4(X)$ are both positive-definite), so h is injective. By theorem 4.2, $H^4(X)$ has rank $4(1+1) = 8$; it follows that the image $h(\Gamma_8)$ is a full-rank sublattice of $H^4(X)$. By lemma 2.2, Γ_8 is unimodular and so, because h is a morphism of lattices, $h(\Gamma_8)$ is again unimodular. Therefore by corollary 2.1 $H^4(X) = h(\Gamma_8)$ and h is surjective. Therefore Γ_8 and $H^4(X)$ are isomorphic lattices, with one isomorphism given by h. □

5.3.2 The general case

This section is devoted to showing that slight generalizations of the arguments and constructions of the previous section work in the general case. For this section, $X = X(2,2)$ is a smooth complete intersection of two quadrics in $\mathbb{P}^{4k+2}$ $(k \geq 1)$.

Incidence correspondence

Proceeding as before, let Z denote the incidence correspondence

$$Z := \{(\Lambda, Q\} : \Lambda \subset Q\} \subset \mathbb{G}(2k, 4k+2) \times \mathbb{P}^N,$$

with $N = \binom{4k+4}{2} - 1$. Let π_1 and π_2 denote the restrictions to Z of the projections of $\mathbb{G}(2k, 4k+2) \times \mathbb{P}^N$ to $\mathbb{G}(2k, 4k+2)$ and $\mathbb{P}^N$, respectively. By the same argument as in the $k = 1$ case, each fibre of π_1 over $\mathbb{G}(2k, 4k+2)$ is biregular to $\mathbb{P}^m$, where $m := (\binom{4k+4}{2} - \binom{2k+2}{2}) - 1 = N - \binom{2k+2}{2}$. The dimension of the Grassmannian $\mathbb{G}(2k, 4k+2)$ is $(2k+1)(2k+2) = 2\binom{2k+2}{2}$, hence Z is irreducible of dimension $N + \binom{2k+2}{2}$ by upper-semicontinuity of fibre dimension and its corollary 5.1. By the same theorem, since the projection of Z to $\mathbb{P}^N$ is surjective, we find that a fibre over a generic quadric $Q \in \mathbb{P}^N$ is $\binom{2k+2}{2}$-dimensional. It follows by the same arguments as before that the fibre over every *smooth* Q is $\binom{2k+2}{2}$-dimensional. Write $\Gamma_Q := \{\Lambda \in \mathbb{G}(2k, 4k+2) : \Lambda \subset Q\} = \pi_1(\pi_2^{-1}(Q))$. Then the Poincaré dual class $[\Gamma_Q]$ is a class in the middle cohomology $H^{2\binom{2k+2}{2}}(\mathbb{G}(2k, 4k+2), \mathbb{Z})$.

There is at least one $2k$-plane contained in X

We need to show that for any smooth quadrics Q, Q' in $\mathbb{P}^{4k+2}$ the cup product $[\Gamma_Q]\cdot[\Gamma_{Q'}]$ is positive. As before, the class $\sigma_{1,2,\ldots,2k+1}$ is self-dual, so it is enough to show that $[\Gamma_Q]\cdot\sigma_{1,2,\ldots,2k+1} > 0$. Let $\mathcal{F}$ be a choice of complete projective flag in $\mathbb{P}^{4k+2}$ such that the intersection of Q with $\mathbb{P}^1$ consists of two distinct points and the intersection of Q with each other element of the flag is smooth. The conditions for a $2k$-plane to be in $\overline{S^{\mathcal{F}}_{1,2,\ldots,2k+1}} \cap \Gamma_Q$ are

0). $\Lambda \subset Q$

1). Λ intersects $\mathbb{P}^1$ in a point

2). Λ intersects $\mathbb{P}^3$ in a line

$\vdots$

2k). Λ intersects $\mathbb{P}^{4k-1}$ in a $\mathbb{P}^{2k-1}$

2k+1). Λ is contained in $\mathbb{P}^{4k+1}$.

We have

Proposition 5.3. *For $k \geq 1$, the number of $2k$-planes Λ in $\mathbb{P}^{4k+2}$ satisfying conditions 0). – $2k+1$). above is 2^{2k+1}.*

Proof. We proceed by induction on k, with the base case $k = 1$ proved as part of lemma 5.3. As in the case $k = 1$, $Q \cap \mathbb{P}^1$ consists of two points, one of which must be contained in Λ. Let p be this point. Further, $Q \cap \mathbb{P}^3$ is smooth, hence contains exactly two lines passing through p, one of which must be contained in Λ. Denote this line by ℓ.

Now let V be the $(4k+3)$-dimensional vector space corresponding to $\mathbb{P}^{4k+2}$, W the 2-dimensional subspace of V corresponding to the line ℓ and let F be the nondegenerate symmetric bilinear form defining the quadric $Q \subset \mathbb{P}^{4k+2}$. A choice of $2k$-plane Λ containing ℓ and contained in Q corresponds to a $(2k+1)$-dimensional subspace P of V containing

W such that $F|_P \equiv 0$. The $(2k+1)$-dimensional subspaces of V containing W correspond to $(2k-1)$-dimensional subspaces of V/W. Let $W^\perp := \{v \in V : F(v,w) = 0 \text{ for all } w \in W\}$. Because F is nondegenerate, $W^\perp$ is $(4k+1)$-dimensional, hence $W^\perp/W$ is a $(4k-1)$-dimensional subspace of V/W. Then F descends to a well-defined nondegenerate bilinear form on $W^\perp/W$, hence defines a smooth quadric $Q' \subset \mathbb{P}(W^\perp/W) \cong \mathbb{P}^{4(k-1)+2}$.

Hence, each $2(k-1)$-plane Λ' contained in $Q' \subset \mathbb{P}(W^\perp/W) \cong \mathbb{P}^{4(k-1)+2}$ corresponds to a $2k$-plane Λ in $\mathbb{P}^{4k+2}$ containing p, ℓ (fixed above) and contained in Q. Moreover, one checks that this correspondence is 1-1 (for example by an argument by contradiction as in the end of the proof of lemma 5.3).

Letting $0')$ stand for the condition $\Lambda' \subset Q'$, we see that Λ satisfies $0), 1)$ and $2)$ (with p and ℓ fixed in the beginning of the argument) if and only if Λ' satisfies $0')$.

We now show that conditions $3). - 2k+1)$ on Λ have natural equivalents for Λ'.

Label the vector spaces composing the flag $\mathcal{F}$ as follows:

$$V_1 \subset V_2 \subset \cdots \subset V_{4k+2} \subset V_{4k+3}.$$

By the choice of flag, the intersection $\mathbb{P}(V_m) \cap Q$ is smooth, hence induces a nondegenerate symmetric bilinear form F_m on V_m by restriction. From now on, we omit the subscript on F_m. For $m \geq 4$, $W \subset V_m$ and we define $W_m^\perp := \{v \in V_m : F(v,w) = 0 \text{ for all } w \in W\}$. The dimension of $W_m^\perp$ is $m-2$ and we have $W_m^\perp = V_m \cap W_{4k+3}^\perp$. Hence we may form the complete flag

$$W_4^\perp/W \subset W_5^\perp/W \subset \cdots \subset W_{4k+2}^\perp/W \subset W_{4k+3}^\perp/W.$$

Projectivizing gives a complete projective flag in $\mathbb{P}^{4(k-1)+2}$.

Let P continue denoting the $(2k+1)$-dimensional vector space corresponding to Λ. Now condition $n)$ for $n \geq 3$ is that P intersects V_{2n-1} along a n-dimensional subspace S_{2n-1}. We have $W \subset S_{2n-1} \subset P$, so the restriction of F to V_{2n-1} must vanish identically on S_{2n-1}. Arguing as before, this implies that necessarily $S_{2n-1} \subset W_{2n-1}^\perp$, so that the $2(k-1)$-plane Λ' intersects $\mathbb{P}(W_{2n-1}^\perp/W) \cong \mathbb{P}^{2(n-2)-1}$ along $\mathbb{P}(S_{2n-1}/W) \cong \mathbb{P}^{(n-2)-1}$ for $n = 3, \ldots, 2k+1$, or, relabeling, Λ' intersects $\mathbb{P}^{2n'-1}$ along a $\mathbb{P}^{n'-1}$ for $n' = 1, \ldots, 2k-1$. By the induction hypothesis, there are $2^{2(k-1)+1}$ planes Λ' satisfying the conditions above. Finally, taking into account the four possibilities for the choice of p and ℓ made in the beginning of the argument, we find that there are a total of 2^{2k+1} planes Λ satisfying the conditions 0) – $2k+1$), as desired. $\square$

Identification of the lattice

The pairwise cup products of the classes $P, \Lambda_0, \ldots, \Lambda_{4k+3}$ are

$$\begin{aligned}
[P]\cdot[P] &= (-1)^{2k}\left\lfloor 1+\frac{2k}{2}\right\rfloor = k+1, \\
[\Lambda_i]\cdot[\Lambda_i] &= (-1)^{2k}\left\lfloor 1+\frac{2k}{2}\right\rfloor = k+1, \qquad i = 0,\ldots,4k+3, \\
[P]\cdot[\Lambda_i] &= (-1)^{2k-1}\left\lfloor 1+\frac{2k-1}{2}\right\rfloor = -k, \qquad i = 0,\ldots,4k+3, \\
[\Lambda_i]\cdot[\Lambda_j] &= (-1)^{2k-2}\left\lfloor 1+\frac{2k-2}{2}\right\rfloor = k, \qquad i,j = 0,\ldots,4k+3, \quad i \neq j.
\end{aligned}$$

Let $H \subset X$ be a hyperplane section. Then

$$\begin{aligned}
[H]^{2k}\cdot[P] &= \deg(P) = 1, \\
[H]^{2k}\cdot[\Lambda_i] &= \deg(\Lambda_i) = 1, \qquad i = 0,\ldots,4k+3, \\
[H]^{2k}\cdot[H]^{2k} &= \deg(X) = \deg(Q_1)\deg(Q_2) = 4.
\end{aligned}$$

Let $\gamma_1, \ldots, \gamma_{4k+5}$ be the generators of $\Gamma_{4(k+1)}$ from lemma 2.3. Let

$$g_i := ([\Lambda_{i-1}] - [\Lambda_i])\,,\ i = 1,\ldots,4k+2,\ g_{4k+3} := \Big([P] - [H]^{2k} + [\Lambda_{4k+2}]\Big),\ g_{4k+4} := [H]^{2k} \text{ and } g_{4k+5} := [P]\,.$$

One verifies that $\gamma_i\cdot\gamma_j = g_i \cup g_j$ for all $i,j = 1,\ldots,4k+5$, hence obtaining an embedding of lattices $h_k : \Gamma_{4(k+1)} \to H^{4k}(X)$, as before (the fact that h_k is an embedding follows from the fact that both lattices are positive-definite). Then theorem 4.2 gives that the rank of $H^{4k}(X(2,2))$ is $4(k+1)$, so that $h_k(\Gamma_{4(k+1)})$ embeds as a full-rank sublattice of $H^{4k}(X)$.

Since $\Gamma_{4(k+1)}$ is unimodular, this shows that the map h_k is an isomorphism. We have

Theorem 5.4. *The cup product lattice of a complete intersection of two smooth quadrics in $\mathbb{P}^{4k+2}$ is the $\Gamma_{4(k+1)}$ lattice.*

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