1. Solve the following equations using the Laplace transform method.

i) $\dfrac{dy}{dt} - y = e^t, \quad y(0) = 1.$

ii) $\dfrac{d^2y}{dt^2} + 4\dfrac{dy}{dt} + 4y = 0, \qquad y(0) = 1, \quad \dfrac{dy}{dt}(0) = 1.$

iii) $\dfrac{d^2y}{dt^2} - 2\dfrac{dy}{dt} + 5y = 0, \qquad y(0) = 2, \quad \dfrac{dy}{dt}(0) = 4.$

iv) $\dfrac{d^3y}{dt^3} - 2\dfrac{d^2y}{dt^2} + \dfrac{dy}{dt} = 2e^t + 2t, \qquad y(0) = 0, \quad \dfrac{dy}{dt}(0) = 0, \quad \dfrac{d^2y}{dt^2}(0) = 0.$

Optional Problem. Also solve the two equations from Problem Set 6 using the Laplace transform:

v) $\dfrac{d^2y}{dt^2} + \dfrac{dy}{dt} - 2y = 2t, \qquad y(0) = 0, \quad \dfrac{dy}{dt}(0) = 1.$

vi) $\dfrac{d^2y}{dt^2} + 16y = t^2 + \sin(4t), \qquad y(0) = \dfrac{127}{128}, \quad \dfrac{dy}{dt}(0) = \dfrac{7}{8}.$

These will require more algebraic manipulations to get the expression for $\mathscr{L}[y](s)$ into a form where the inverse transform can be found by inspection.

2. i) Show that $\int_0^\infty f(t)e^{-st}\,dt$ converges for $s = s_0 - \sigma$ if and only if $\int_0^\infty \left(e^{\sigma t} f(t)\right) e^{-st}\,dt$ converges for $s = s_0$.

Hence, show that if the domain of $\mathscr{L}[f]$ is all $s > a$, then the domain of $\mathscr{L}[e^{\sigma t} f]$ is all $s > a + \sigma$, and we have

$$\mathscr{L}[e^{\sigma t} f](s) = \mathscr{L}[f](s - \sigma) \quad \text{for all} \quad s > a + \sigma.$$

ii) Taking the four transforms

$$\mathscr{L}[\cos(\omega t)](s) = \frac{s}{s^2 + \omega^2},\ s > 0, \qquad \mathscr{L}[\sin(\omega t)](s) = \frac{\omega}{s^2 + \omega^2},\ s > 0,$$

$$\mathscr{L}[t\cos(\omega t)](s) = \frac{s^2 - \omega^2}{(s^2 + \omega^2)^2},\ s > 0 \qquad \text{and} \qquad \mathscr{L}[t\sin(\omega t)](s) = \frac{2\omega s}{(s^2 + \omega^2)^2},\ s > 0$$

as known, find

$$\mathscr{L}[e^{\sigma t}\cos(\omega t)], \quad \mathscr{L}[te^{\sigma t}\cos(\omega t)], \quad \mathscr{L}[e^{\sigma t}\sin(\omega t)] \quad \text{and} \quad \mathscr{L}[te^{\sigma t}\cos(\omega t)].$$

3. As a reminder, earlier in the term we defined the functions

$$\cosh(\omega t) = \frac{e^{\omega t} + e^{-\omega t}}{2} \qquad \text{and} \qquad \sinh(\omega t) = \frac{e^{\omega t} - e^{-\omega t}}{2}, \qquad t \in \mathbb{R}.$$

Find expressions for $\mathscr{L}[\cosh(\omega t)]$ and $\mathscr{L}[\sinh(\omega t)]$ in the following two ways:

i) Take as known the transform

$$\mathscr{L}[e^{\sigma t}](s) = \frac{1}{s-\sigma}, \quad s > 0,$$

and use linearity of $\mathscr{L}$.

ii) Check that

$$\cosh(0) = 1, \qquad \frac{d}{dt}\cosh(\omega t)\bigg|_{t=0} = 0,$$

and

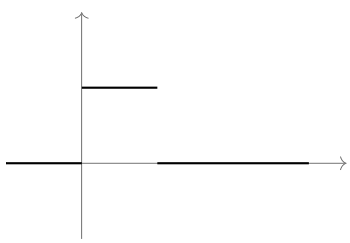
$$\sinh(0) = 0, \qquad \frac{d}{dt}\sinh(\omega t)\bigg|_{t=0} = \omega.$$

Then, find a linear homogeneous differential equation with constant coefficients that has cosh and sinh as solutions, apply $\mathscr{L}$ to both sides of the differential equation, and use the initial conditions found above.

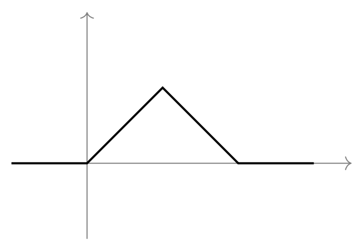
(Although you do not have to show this, the domain of both $\mathscr{L}[\cosh(\omega t)]$ and $\mathscr{L}[\sinh(\omega t)]$ is equal to $s > |\omega|$, as may be tempting to guess from the domains of $e^{\omega t}$ and $e^{-\omega t}$.)

Optional Problem. Compute $\mathscr{L}[\cosh(\omega t)]$ and $\mathscr{L}[\sinh(\omega t)]$ from the definition, and verify the claim just made about their domains.

4. Let f and g denote the following two functions.



and



$$f(t) = \begin{cases} 1 & 0 \le t \le 1 \\ 0 & \text{otherwise} \end{cases} \qquad g(t) = \begin{cases} t & 0 \le t \le 1 \\ 2-t & 1 \le t \le 2 \\ 0 & \text{otherwise} \end{cases}$$

i) Show that the convolution $f * f$ of f with itself is equal to g, in the following two ways.

First, compute $f * f$ using the definition of convolution

$$f * f(t) = \int_0^t f(u)f(t-u)\,du$$

and check that the result is equal to g.

Second, write both f and g as an expression involving step functions, and apply the property $\mathscr{L}[f * f] = \mathscr{L}[f]\mathscr{L}[f]$ of the Laplace transform.

ii) Compute the convolution $(2f) * g$ of $2f$ with g in the same two ways as part i).

Optional Problem. Investigate the motion of a simple harmonic oscillator starting from rest

$$\frac{d^2y}{dt^2} + y = F(t), \qquad y(0) = 0, \quad \frac{dy}{dt}(0) = 0,$$

subject to forcing functions $f(t)$, $g(t)$ and $((2f) * g)(t)$.