1. Solve the following equations using the method of variation of parameters.

i) $\dfrac{dy}{dt} - \dfrac{2t}{t^2+1}y = 1, \quad y(0) = 0.$ (The integral $\int \frac{ds}{s^2+1} = \arctan(s) + C$ may be useful.)

ii) $\dfrac{d^2y}{dt^2} + 2\dfrac{dy}{dt} + y = \dfrac{e^{-t}}{t}, \quad t > 0 \qquad y(1) = 0,\ \dfrac{dy}{dt}(1) = -e^{-1}.$

iii) $2t^2\dfrac{d^2y}{dt^2} + 3t\dfrac{dy}{dt} - y = \dfrac{1}{t}, \quad t > 0, \qquad y(1) = 0,\ \dfrac{dy}{dt}(1) = \dfrac{11}{6},$
given that $\phi_1 = t^{1/2}$ and $\phi_2 = t^{-1}$ are solutions of the associated homogeneous equation.

2. A first-order differential equation of the form

$$\frac{dy}{dt} + a(t)y = F(t)y^k, \quad k \neq 0, 1 \text{ real}, \quad y \neq 0 \tag{1}$$

is called a *Bernoulli equation.*

Although Bernoulli equations are nonlinear (unless $k = 0$ or $k = 1$, which is the reason for excluding these exponents), they can always be converted to linear equations by a change of variable.

i) Let $y(t)$ be a solution of the Bernoulli equation (1). Let $v(t) = y(t)^{1-k}$. Show that $\frac{1}{1-k}\frac{dv}{dt} = y^{-k}\frac{dy}{dt}$. Then, show that $v(t)$ is a solution of the differential equation

$$\frac{dv}{dt} + (1-k)a(t)v = (1-k)F(t), \tag{2}$$

which is now linear.

(Conversely, one can similarly show that if $v(t)$ is a solution of (2), then $y(t) = v(t)^{1/(1-k)}$ is a solution of (1).)

ii) Solve the equation

$$\frac{dy}{dt} = \epsilon y - \sigma y^3, \quad \epsilon > 0,\ \sigma > 0, \quad y(0) = \sqrt{2\epsilon/\sigma}$$

by recognizing it as a Bernoulli equation and making the above change of variable.

iii) Solve the equation

$$\frac{dy}{dt} + y = ty^3, \quad y(0) = 1.$$

3. In this problem, we look at resonance in a simple harmonic oscillator.

Consider an undamped spring with spring constant k hanging vertically, with one end fixed and the other end attached to a mass m. As derived in Problem Set 04, the equation of motion of the mass about its rest point is

$$\frac{d^2y}{dt^2} + \omega_0^2 y = 0,$$

where ω_0 is the frequency of the two solutions, $\cos(\omega_0 t)$ and $\sin(\omega_0 t)$, called the *natural frequency* of the oscillator.

Suppose that we introduce an oscillating driving force $F(t) = mF_0 \cos(\omega t)$, with amplitude (mF_0) and frequency ω. The equation of motion then becomes

$$\frac{d^2y}{dt^2} + \omega_0^2 y = F_0 \cos(\omega t). \tag{3}$$

Suppose that $\omega = \omega_0$, so that the frequency of the driving force exactly matches the natural frequency of the oscillator.

i) Show that the set of solutions of equation (3) is equal to

$$\left\{ \frac{F_0}{2\omega_0} t \sin(\omega_0 t) + b_1 \cos(\omega_0 t) + b_2 \sin(\omega_0 t) : b_1,\, b_2 \in \mathbb{R} \right\}.$$

ii) Suppose that the oscillator starts at rest, so that we have the initial conditions

$$y(0) = 0, \quad \frac{dy}{dt}(0) = 0.$$

Find the solution of equation (3) satisfying these initial conditions and sketch the graph of the solution for $t > 0$.

iii) Suppose that the initial conditions are instead

$$y(0) = 0, \quad \frac{dy}{dt}(0) = 10\omega_0.$$

Find the solution of equation (3) satisfying these initial conditions and sketch the graph of the solution for $t > 0$.

4. Let $c_1, \ldots, c_r$ be a collection of real numbers.

i) Suppose that the numbers $c_1, \ldots, c_r$ are pairwise distinct (meaning $c_i \neq c_j$ if $i \neq j$). Find a linear homogeneous equation whose space of solutions has the set $\{e^{c_1 t}, \ldots, e^{c_r t}\}$ as a basis. Using results from class, conclude that the Wronskian $W(e^{c_1 t}, \ldots, e^{c_r t})(t)$ is not equal to zero for all $t \in \mathbb{R}$.

ii) Conclude that the *Vandermonde determinant* of $c_1, \ldots, c_r$,

$$\det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ c_1 & c_2 & \cdots & c_r \\ c_1^2 & c_2^2 & \cdots & c_r^2 \\ \vdots & \vdots & \ddots & \vdots \\ c_1^{r-1} & c_2^{r-1} & \cdots & c_r^{r-1} \end{pmatrix}$$

is not equal to zero if and only if the numbers $c_1, \ldots, c_r$ are pairwise distinct. (*Reminder:* in proving an ‘if and only if’ statement, there two directions of implication to show. One of the directions here is fairly simple, while the other can be proved using the result of part i).)

The Vandermonde determinant comes up often throughout Mathematics, and the property proved in part ii) is very useful to know.