1. Solve the differential equation

$$\frac{d^2y}{dt^2} + \frac{dy}{dt} - 2y = 2t, \quad y(0) = 0, \quad \frac{dy}{dt}(0) = 1 \tag{1}$$

using the method of undetermined coefficients, as follows:

i) Find an annihilator $p_F\left(\frac{d}{dt}\right)$ of the function $F(t) = 2t$. That is, find a polynomial differential operator $p_F\left(\frac{d}{dt}\right)$ with constant coefficients such that

$$p_F\left(\frac{d}{dt}\right)(2t) = 0.$$

ii) Find a basis of solutions of the homogeneous linear differential equation

$$\left[p_F\left(\frac{d}{dt}\right)\left(\frac{d^2}{dt^2} + \frac{d}{dt} - 2\right)\right]y = 0.$$

iii) Find a particular solution ϕ_p of (1) in the span of the basis elements found in part ii).
(For this step, it helps save some work to disregard the basis elements that are solutions of the homogeneous linear equation $\frac{d^2y}{dt^2} + \frac{dy}{dt} - 2y = 0$ associated to (1).)

iv) Find a solution of (1) satisfying the given initial conditions.
(As a reminder, we have shown in lecture that the set of solutions of (1) is equal to
$\left\{\phi_p + \phi_h : \phi_h \text{ is a solution of the homogeneous linear equation } \frac{d^2y}{dt^2} + \frac{dy}{dt} - 2y = 0 \text{ associated to (1)}\right\}$.)

2. i) Suppose that for $k = 1, \dots, n$, the function $\phi_k : I \to \mathbb{R}$ is a solution of the differential equation

$$\frac{d^ry}{dt^r} + a_{r-1}(t)\frac{d^{r-1}y}{dt^{r-1}} + \cdots + a_1(t)\frac{dy}{dt} + a_0(t)y = F_k(t).$$

Show that then the sum $\phi = \phi_1 + \cdots + \phi_n$ is a solution of the equation

$$\frac{d^ry}{dt^r} + a_{r-1}(t)\frac{d^{r-1}y}{dt^{r-1}} + \cdots + a_1(t)\frac{dy}{dt} + a_0(t)y = F_1(t) + \cdots + F_n(t).$$

ii) Suppose that for $k = 1, \dots, n$, the polynomial operator $p_{F_k}\left(\frac{d}{dt}\right)$ with constant coefficients is an annihilator of the function F_k over I. Show that then the product

$$p_{F_1}\left(\frac{d}{dt}\right)p_{F_2}\left(\frac{d}{dt}\right)\cdots p_{F_n}\left(\frac{d}{dt}\right)$$

is an annihilator of the function $F_1 + \cdots + F_n$ over I.

3. Solve the differential equation

$$\frac{d^2y}{dt^2} + 16y = t^2 + 2\cos(2t)\sin(2t), \qquad y(0) = \frac{127}{128}, \quad \frac{dy}{dt}(0) = \frac{7}{8}$$

using the method of undetermined coefficients.
(Using the sine angle addition identity, we have $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$. To handle the right-hand side, it may help to apply either of the results of Problem 2.)