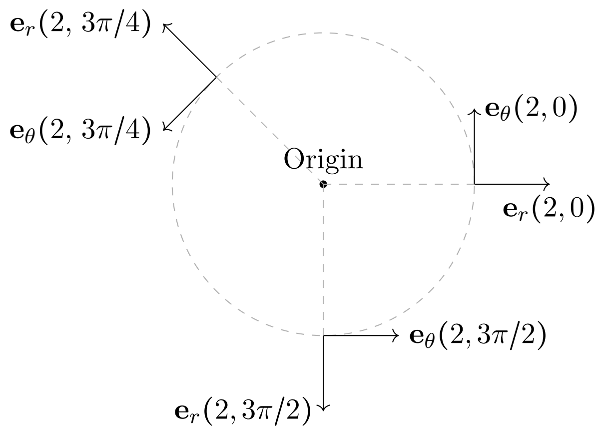
Velocity and Acceleration in Polar Coordinates

The Argument (r, θ) of $\mathbf{e}_r$ and $\mathbf{e}_\theta$. What does the pair (r, θ) refer to in the notation $\mathbf{e}_r(r, \theta)$ and $\mathbf{e}_\theta(r, \theta)$?

The main difference between the familiar direction vectors $\mathbf{e}_x$ and $\mathbf{e}_y$ in Cartesian coordinates and the polar direction vectors is that the polar direction vectors change depending on where they are relative to the origin.

To specify the direction vector it is therefore necessary to give its 'address' in $\mathbb{R}^2$. The pair (r, θ) can be thought of as that address. It is not a description of the direction vector itself, but rather the location of the base point (or tail) of the direction vector.

Here are a few examples of direction vector pairs:



Stated a little differently, $\mathbf{e}_r$ and $\mathbf{e}_\theta$ are really *vector fields* (there is slight subtlety at the origin — the vector field has many vectors at this point).

Velocity and Acceleration. Given a vector in $\mathbb{R}^2$, its decomposition into $\mathbf{e}_r(r, \theta)$ and $\mathbf{e}_\theta(r, \theta)$ components will be different depending on what (r, θ) is.

For example, the vector $\mathbf{e}_x + \mathbf{e}_y$ (that has Cartesian coordinates $(x, y) = (1, 1)$) decomposes as follows in terms of the direction vector pairs drawn above:

$$\mathbf{e}_x + \mathbf{e}_\mathbf{y} = \mathbf{e}_r(2, 0) + \mathbf{e}_\theta(2, 0),$$
$$\mathbf{e}_x + \mathbf{e}_\mathbf{y} = -\sqrt{2}\,\mathbf{e}_\theta(2, 3\pi/4),$$
$$\mathbf{e}_x + \mathbf{e}_\mathbf{y} = -\mathbf{e}_r(2, 3\pi/2) + \mathbf{e}_\theta(2, 3\pi/2).$$

In the expression for the velocity in polar coordinates,

$$\mathbf{v}(t) = \frac{dr}{dt}\,\mathbf{e}_r(r(t), \theta(t)) + r\,\frac{d\theta}{dt}\,\mathbf{e}_\theta(r(t), \theta(t)),$$

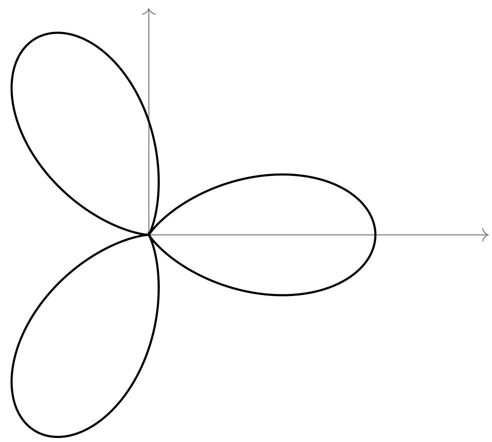
for each t, the vector $\mathbf{v}(t)$ is decomposed into $\mathbf{e}_r(r(t), \theta(t))$ and $\mathbf{e}_\theta(r(t), \theta(t))$-components. In general, the direction vectors will vary with t.

Similar remarks hold for the decomposition of the acceleration $\mathbf{a}(t)$.

Example. Consider the path parametrized in polar coordinates by

$$t \mapsto (1 + \cos(3t),\, t),\ \ t \in [0, 2\pi].$$

This is the three-leafed path we have seen in lecture. It looks like:



The Curve $r = 1 + \cos(3\theta)$

We compute that

$$\frac{dr}{dt} = -3\sin(3t), \qquad \frac{d\theta}{dt} = 1,$$

$$\frac{d^2r}{dt^2} = -9\cos(3t), \quad \frac{d^2\theta}{dt^2} = 0.$$

Therefore, the velocity is

$$\begin{aligned}\mathbf{v}(t) &= (-3\sin(3t))\ \mathbf{e}_r(1+\cos(3t),\, t) + (1+\cos(3t))\cdot 1\ \mathbf{e}_\theta(1+\cos(3t),\, t)\\ &= -3\sin(3t)\,\mathbf{e}_r(1+\cos(3t),\, t) + (1+\cos(3t))\,\mathbf{e}_\theta(1+\cos(3t)\,, 1t).\end{aligned}$$

and the acceleration, using the general expression

$$\mathbf{a}(t) = \left(\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right)\mathbf{e}_r(r(t),\theta(t)) + \left(r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt}\right)\mathbf{e}_\theta(r(t),\theta(t)),$$

is

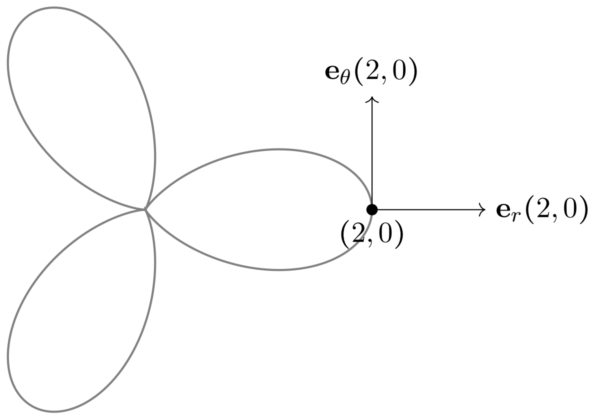
$$\begin{aligned}\mathbf{a}(t) &= \left(-9\cos(3t) - (1+\cos(3t))\cdot 1^2\right)\ \mathbf{e}_r(1+\cos(3t),\, t) + (0 + 2(-3\sin(3t))\cdot 1)\,\mathbf{e}_\theta(1+\cos(3t),\, t)\\ &= -\left(1 + 10\cos(3t)\right)\ \mathbf{e}_r(1+\cos(3t),\, t) - 6\sin(3t)\,\mathbf{e}_\theta(1+\cos(3t),\, t).\end{aligned}$$

Now, let's plot the velocity and acceleration vectors for a few values of t.

$t = 0$**.** The position is

$$\mathbf{r}(0) = (1+\cos(0),\, 0) = (2, 0).$$

The direction vectors look as follows at this point:



The velocity is

$$\mathbf{v}(0) = -3\sin(0)\,\mathbf{e}_r(2,0) + (1+\cos(0))\mathbf{e}_\theta(2,0) = 2\,\mathbf{e}_\theta(2,0)$$

and the acceleration is

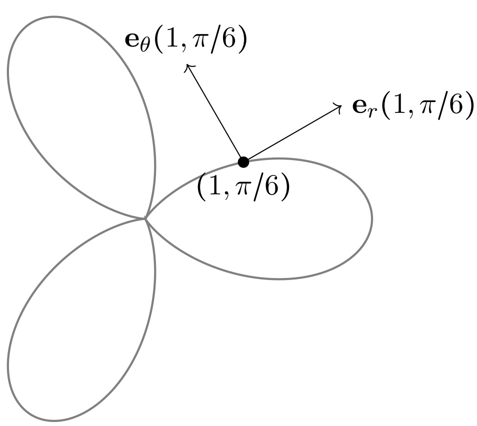
$$-(1+10\cos(0))\,\mathbf{e}_r(2,0) - 6\sin(0)\,\mathbf{e}_\theta(2,0) = -11\,\mathbf{e}_r(2,0).$$

Sketching the vectors, we have

$t = \pi/6$**.** Notice that $3\cdot\pi/6 = \pi/2$ (this is useful to compute because the cosines and sines have threes in their arguments). The position is

$$\mathbf{r}(\pi/6) = (1+\cos(\pi/2),\, \pi/6) = (1, \pi/6).$$

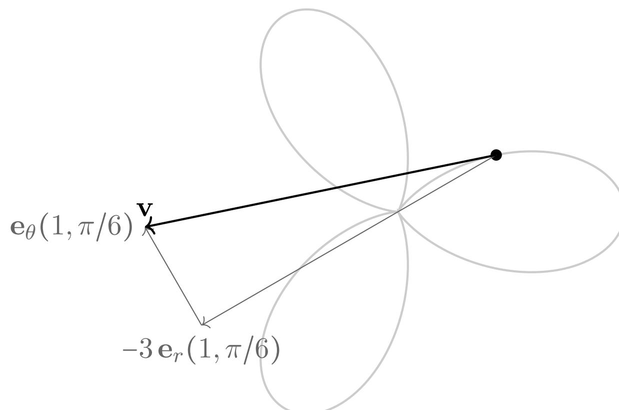
The direction vectors look like:



The velocity is

$$\mathbf{v}(\pi/6) = -3\sin(\pi/2)\,\mathbf{e}_r(1,\pi/6) + (1+\cos(\pi/2))\,\mathbf{e}_\theta(1,\pi/6) = -3\,\mathbf{e}_r(1,\pi/6) + \mathbf{e}_\theta(1,\pi/6).$$

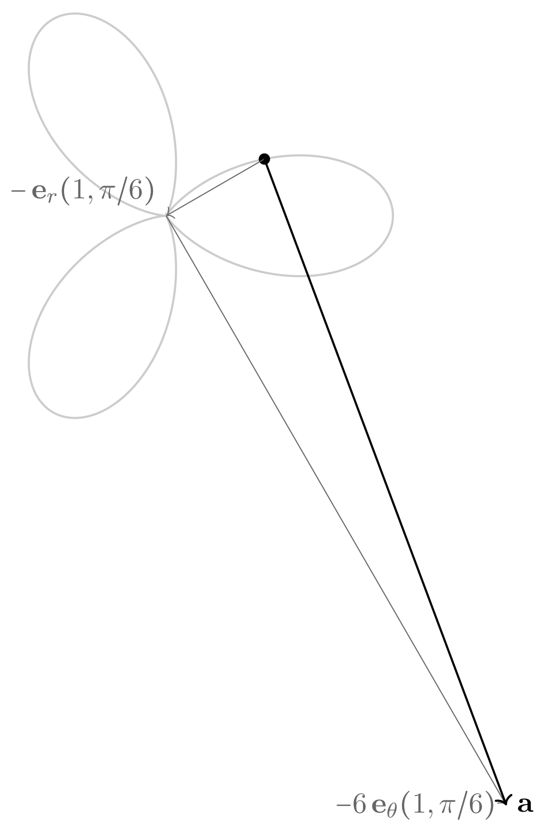
Adding tip-to-tail, this looks like



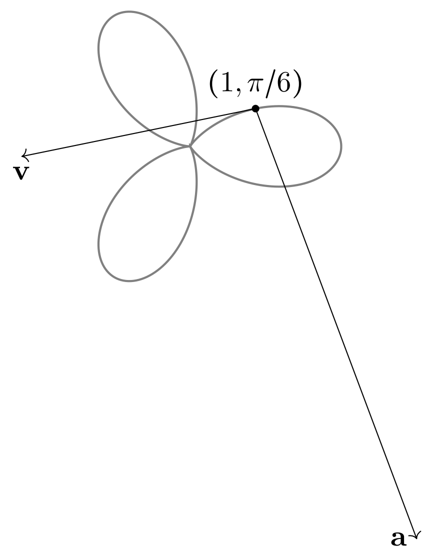
The acceleration is

$$\mathbf{a}(\pi/6) = -(1+10\cos(\pi/2))\,\mathbf{e}_r(1,\pi/6) - 6\sin(\pi/2)\,\mathbf{e}_\theta(1,\pi/6) = -1\,\mathbf{e}_r(1,\pi/6) - 6\,\mathbf{e}_\theta(1,\pi/6).$$

Adding tip-to-tail, this looks like



So that the full picture is

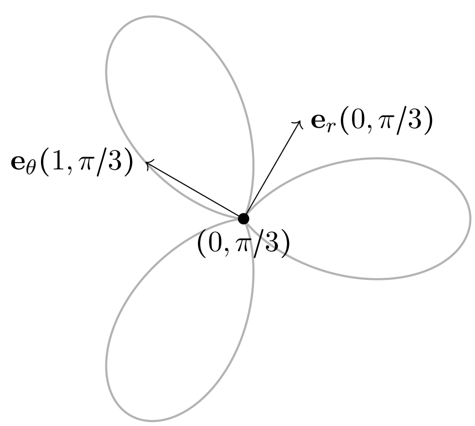


$t = \pi/3$**.** This is an interesting one, because the parametrization comes to a rest at the origin. We have $3 \cdot \pi/3 = \pi$.

The position is

$$\mathbf{r}(\pi/3) = (1 + \cos(\pi), \pi/3) = (0, \pi/3).$$

The direction vectors look like



The velocity is

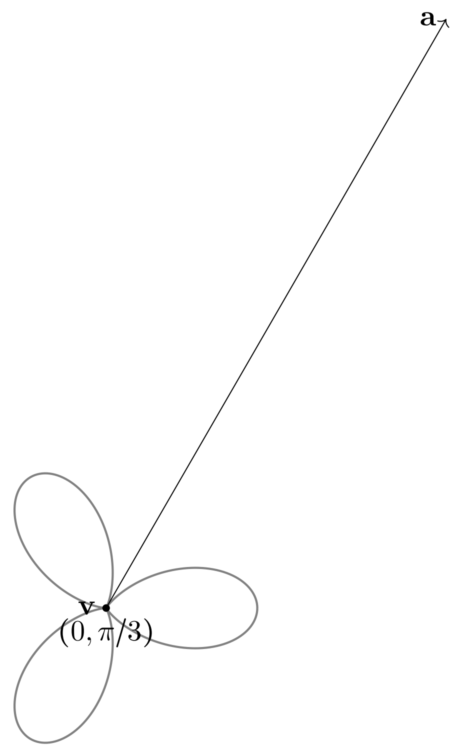
$$\mathbf{v}(\pi/3) = -3\sin(\pi)\,\mathbf{e}_r(0, \pi/3) + (1 + \cos(\pi))\,\mathbf{e}_\theta(0, \pi/3) = 0\,\mathbf{e}_r(0, \pi/3) + 0\,\mathbf{e}_\theta(0, \pi/3).$$

The particle is at rest!

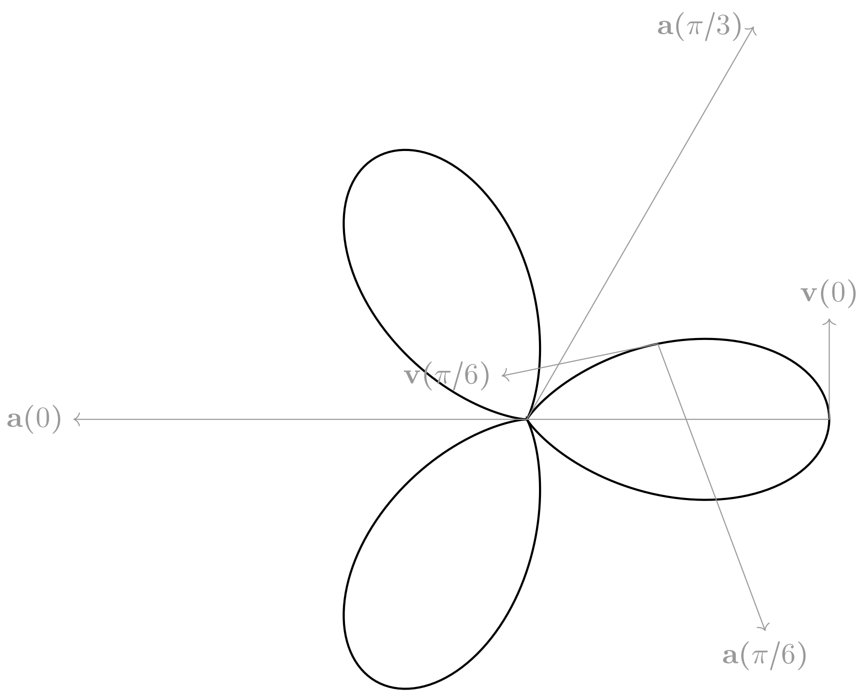
The acceleration is

$$\mathbf{a}(\pi/3) = -(1 + 10\cos(\pi))\,\mathbf{e}_r(0, \pi/3) - 6\sin(\pi)\,\mathbf{e}_\theta(0, \pi/3) = 9\,\mathbf{e}_r(0, \pi/3).$$

The full picture is



Finally, here are scaled velocity and acceleration vectors on the same picture (they have the correct lengths relative to each other, but each has been scaled down by the same amount):



Dot Products. Because $\mathbf{e}_r(r,\theta)$ and $\mathbf{e}_\theta(r,\theta)$ are orthogonal, *as long as both are at the same* (r,θ), we can take dot products as usual:

$$(R_1\mathbf{e}_r(r,\ \theta) + \Theta_1\mathbf{e}_\theta(r,\theta)) \cdot (R_2\mathbf{e}_r(r,\ \theta) + \Theta_2\mathbf{e}_\theta(r,\theta)) = R_1R_2 + \Theta_1\Theta_2.$$

Therefore, we find that, in general,

$$\|\mathbf{v}(t)\| = \sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2}, \quad \text{and}$$

$$\|\mathbf{a}(t)\| = \sqrt{\left(\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right)^2 + \left(r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt}\right)^2}.$$

For the example $r = 1 + \cos(3\theta)$, the velocity and acceleration are, once again,

$$\begin{aligned}\mathbf{v}(t) &= -3\sin(3t)\,\mathbf{e}_r(1+\cos(3t),\ t) + (1+\cos(3t))\,\mathbf{e}_\theta(1+\cos(3t),\ t)\\ \mathbf{a}(t) &= -(1+10\cos(3t))\,\mathbf{e}_r(1+\cos(3t),\ t) - 6\sin(3t)\,\mathbf{e}_\theta(1+\cos(3t),\ t).\end{aligned}$$

So that, taking the dot products,

$$\begin{aligned}\|\mathbf{v}(t)\| &= \sqrt{9\sin^2(3t) + (1+\cos(3t))^2} = \sqrt{8\sin^2(3t) + 2\cos(3t) + 2},\\ \|\mathbf{a}(t)\| &= \sqrt{(1+10\cos(3t))^2 + 36\sin^2(3t)} = \sqrt{64\cos^2(3t) + 20\cos(3t) + 37}.\end{aligned}$$