MTHE 227 Problem Set 9

Due Thursday November 17 2016 at the beginning of class

1 (Cross-Product in $\mathbb{R}^2$ and $\mathbb{R}^3$)**.** For this problem, to help distinguish between the cross-products in 2- and 3-space, for vectors

$$\mathbf{v_1} = (x_1, y_1),\ \mathbf{v_2} = (x_2, y_2) \text{ in } \mathbb{R}^2 \quad \text{and} \quad \mathbf{w_1} = (x_1, y_1, z_1),\ \mathbf{w_2} = (x_2, y_2, z_2) \text{ in } \mathbb{R}^3,$$

write

$$\operatorname{cross}_2(\mathbf{v_1}, \mathbf{v_2}) = \det\begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix} \qquad \text{and} \qquad \operatorname{cross}_3(\mathbf{w_1}, \mathbf{w_2}) = \det\begin{pmatrix} \mathbf{e_x} & \mathbf{e_y} & \mathbf{e_z} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{pmatrix}.$$

Embed $\mathbb{R}^2_{(x,y)}$ into $\mathbb{R}^3_{(x,y,z)}$ by the map $(x,y) \mapsto (x,y,0)$ (the image being the plane $z = 0$).

(a) Let $\mathbf{v_1}$, $\mathbf{v_2}$ be vectors in $\mathbb{R}^2_{(x,y)}$ and $\mathbf{w_1}$, $\mathbf{w_2}$ their images under the embedding. Check that

$$\operatorname{cross}_2(\mathbf{v_1}, \mathbf{v_2}) = \operatorname{cross}_3(\mathbf{w_1}, \mathbf{w_2}) \cdot \mathbf{e_z}\ .$$

(b) Let $\mathbf{r} : t \mapsto (x(t), y(t), 0)$, $t \in [a, b]$ be a parametrized path in $\mathbb{R}^3_{(x,y,z)}$ (thought of as the image of a parametrized path in $\mathbb{R}^2_{(x,y)}$ under the above embedding). Denote the velocity vector at time t by $\mathbf{r}'(t) = (x'(t), y'(t), 0)$. Check that

$$\begin{aligned} \mathbf{n}_+(t) &:= (\, y'(t), -x'(t), 0\,) = \operatorname{cross}_3(\mathbf{r}', \mathbf{e_z}) \qquad \text{and} \\ \mathbf{n}_-(t) &:= (\, -y'(t), x'(t), 0\,) = \operatorname{cross}_3(\mathbf{e_z}, \mathbf{r}'). \end{aligned}$$

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Optional Problem (Harder). Embed $\mathbb{R}^2_{(x,y)}$, $\mathbb{R}^2_{(y,z)}$ and $\mathbb{R}^2_{(x,z)}$ into $\mathbb{R}^3_{(x,y,z)}$ as the planes $z = 0$, $x = 0$ and $y = 0$, respectively. Let $\pi_z : \mathbb{R}^3_{(x,y,z)} \to \mathbb{R}^2_{(x,y)}$ be the projection map $(x, y, z) \mapsto (x, y)$, and similarly define π_x, the projection onto $\mathbb{R}^2_{(y,z)}$, and π_y, the projection onto $\mathbb{R}^2_{(x,z)}$.

Let P be a parallelogram in $\mathbb{R}^3$, and denote its images under the above projections by $P_x = \pi_x(P)$, $P_y = \pi_y(P)$ and $P_z = \pi_z(P)$. Show that

$$\operatorname{area}(P) = \sqrt{\operatorname{area}(P_x)^2 + \operatorname{area}(P_y)^2 + \operatorname{area}(P_z)^2}.$$

Conclude, by applying the Cauchy-Schwarz inequality or otherwise, that

$$\operatorname{area}(P) \geq \frac{1}{\sqrt{3}}(\operatorname{area}(P_x)+\operatorname{area}(P_y)+\operatorname{area}(P_z)) = \sqrt{3}\cdot\text{Arithmetic Mean}(\operatorname{area}(P_x), \operatorname{area}(P_y), \operatorname{area}(P_z)).$$

Can you find a P for which equality holds?

2 (Triple Cross Product)**.** Find three vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ in $\mathbb{R}^3$ such that

$$(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} \neq \mathbf{u} \times (\mathbf{v} \times \mathbf{w}).$$

(If you are stuck, there is a suggestion at the end of the problem set. But try to find the vectors yourself — there are many possibilities.)

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Optional Problem (Messy). Show the identity

$$(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{v} \cdot \mathbf{w})\mathbf{u}$$

by expanding out in coordinates, and conclude that

$$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}.$$

Conclude that $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = \mathbf{u} \times (\mathbf{v} \times \mathbf{w})$ if and only if either: $\mathbf{u}$ and $\mathbf{w}$ are both perpendicular to $\mathbf{v}$, or $\mathbf{u} = \lambda \mathbf{w}$ for some $\lambda \in \mathbb{R}$.

Also, conclude that

$$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) + \mathbf{v} \times (\mathbf{w} \times \mathbf{u}) + \mathbf{w} \times (\mathbf{u} \times \mathbf{v}) = \mathbf{0} \qquad \text{(the Jacobi identity).}$$

3 (Examples of Centroids of Curves)**.** In lecture, we learned how to compute the coordinates of the center of mass of a curve C in $\mathbb{R}^3$. When C has uniform unit density (that is, $\delta = 1$), the center of mass of C is also called the *centroid*. The coordinates of the centroid of C are then

$$\frac{1}{\int_C ds}\left(\int_C x\,ds,\ \int_C y\,ds,\ \int_C z\,ds\right).$$

A similar expression is true for a curve in $\mathbb{R}^2$, omitting the z-coordinate.

Find the centroids of the following curves in $\mathbb{R}^2$. You may use symmetry arguments to reduce the number of computations you need to do.

(a) The line segment parametrized by $t \mapsto (t,\, mt)$, $t \in [0, \frac{1}{m}]$, where $m > 0$ is the slope.

(b) The right semicircle $t \mapsto (a\cos(t),\, a\sin(t))$, $t \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ of radius a centered at the origin.

(c) The circle $t \mapsto (b + a\cos(t),\, a\sin(t))$, $t \in [0, 2\pi]$ of radius a centered at $(b, 0)$, with $b > a$ (feel free to write down the answer without computation if you see it).

(d) The piecewise curve $C = C_1 + C_2 + C_3$, where C_1 is the line segment from $(0, b)$ to (a, b), C_2 the line segment from (a, b) to $(a, -b)$, and C_3 the line segment from $(a, -b)$ to $(0, -b)$, where $a > 0$ and $b > 0$. The curve C is a $a \times 2b$ rectangle, with the left side missing.

(e) Find the integral $\int_C x\,ds$ for the parabola segment $t \mapsto (t,\, t^2)$, $t \in [0, 1]$.

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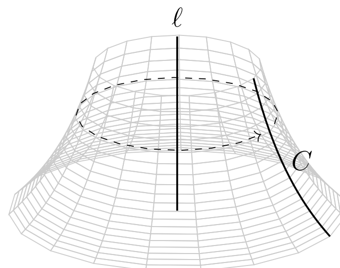
Optional Problem (Harder). Find the coordinates of the centroid of the parabola segment in part (e). The standard approach to the integrals involved uses sinh-substitution (!).

4 (Surfaces of Revolution)**.** A surface of revolution is the surface obtained by rotating a plane curve C about a line ℓ (called the axis of rotation) that is coplanar with C.



To obtain a surface according to the definition in lecture, we require that ℓ does not intersect C, except possibly at the endpoints of C. To obtain a smooth surface (except for at most finitely many nonsmooth curves, which do not affect surface area), we require that there exists a parametrization $t \mapsto \mathbf{r}(t) = (x(t), z(t)),\ t \in [a,b]$ of C with $\mathbf{r}'(t) \neq \mathbf{0}$ for all t (with at most finitely many exceptions).

Suppose that C lies in the xz-plane with $x > 0$, ℓ is the z-axis, and fix a parametrization of C as above.

(a) Find the unit vector that is obtained by rotating $\mathbf{e_x}$ counterclockwise by θ radians about the z-axis.

(b) Using the parametrization $t \mapsto \mathbf{r}(t) = (x(t), z(t)),\ t \in [a,b]$ of C, parametrize the curve obtained by rotating C counterclockwise by θ radians about the z-axis (it will lie in the plane spanned by $\mathbf{e_z}$ and the vector from part (a)). Your parametrization will involve the functions $x(t)$ and $z(t)$.

(c) Parametrize the surface of revolution of C, taking one of the parameters to be the parameter t of C, and the other parameter to be the angle θ. What do the t- and θ-coordinate curves look like?

(d) Find the tangent vectors $\mathbf{T}_t(t,\theta)$ and $\mathbf{T}_\theta(t,\theta)$ at all points.

(e) Find the normal $\mathbf{N}(t,\theta) = \mathbf{T}_t(t,\theta) \times \mathbf{T}_\theta(t,\theta)$ and its magnitude $\|\mathbf{N}(t,\theta)\|$ at all points.

(f) Show that the surface area of the surface of revolution of C is equal to

$$2\pi \int_a^b x(t)\sqrt{x'(t)^2 + z'(t)^2}\, dt = 2\pi \int_C x\, ds.$$

(g) Conclude that the following theorem holds:

Theorem (Pappus)**.** *The surface area of the surface of revolution of a curve C is equal to the product*

$$\text{arclength}(C) \cdot \text{distance travelled by the centroid of } C.$$

(h) For each of the curves in Problem 3, sketch its surface of revolution about the z-axis and find the surface area using Pappus's theorem.

Possibility for 2: $\mathbf{u} = (1,0,0)$, $\mathbf{v} = (1,0,0)$ and $\mathbf{w} = (0,1,0)$