MTHE 227 Problem Set 8

Due Thursday November 10 2016 at the beginning of class

1 (Path-Connected and Simply-Connected)**.** Which of the following spaces are path-connected? Which are simply-connected? (For cases that are not path-connected, draw two points that cannot be joined by a path. For cases that are path- but not simply-connected, draw a simple closed curve (i.e. a loop) that cannot be continuously deformed to a point while staying in the region. For cases that are simply-connected, it is enough to state this (you do not have to justify it).)

(a) $\mathbb{R}^2$ with the circle $x^2 + y^2 = 1$ removed.

(b) $\mathbb{R}^3$ with the circle $x^2 + y^2 = 1,\ z = 0$ removed.

(c) The annulus $\{(x, y) : 1 < x^2 + y^2 < 2\}$ in $\mathbb{R}^2$.

(d) $\mathbb{R}^3$ with a point removed.

(e) $\mathbb{R}^3$ with a line removed.

(f) $\mathbb{R}^3$ with the helix $(\cos t, \sin t, t),\ t \in [0, 4\pi]$ removed.

2 (Curl Test)**.** In lecture, we have shown the following theorem:

Theorem (Curl Test)**.** *Let* $\mathbf{F}(x, y) = (\ F_1(x, y), F_2(x, y)\ )$ *be a vector field defined in a simply-connected region* X*. If*

$$\operatorname{curl} \mathbf{F} := \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 0$$

at every point of X*, then* $\mathbf{F}$ *is conservative.*

Conversely, let $\mathbf{G}(x, y) = (\ G_1(x, y), G_2(x, y)\ )$ *be a vector field defined in* any *region* X *(not necessarily simply-connected). If* $\operatorname{curl} \mathbf{G}(x, y) \neq 0$ *for some point* (x, y) *in* X*, then* $\mathbf{G}$ *is not conservative.*

Applying the curl test, show that the following vector fields defined on $\mathbb{R}^2$ are not conservative.

(a) $(\ x \sin(y^2), y \sin(x^2)\ )$.

(b) $(\ 2x + 3y^2 + 5x^3, 5y + 3x^2 + 2y^3\ )$.

On the other hand, show that the following vector fields defined on $\mathbb{R}^2$ are conservative, again applying the curl test (note: $\mathbb{R}^2$ is simply-connected, so the curl test applies!):

(c) $\left(\ln y + \dfrac{y}{x}, \ \ln x + \dfrac{x}{y} \right)$.

(d) $(\ (1 + xy)\, e^{xy},\ x^2 e^{xy}\ )$.

Remark. We found potential functions for the vector fields of parts (c) and (d) in Problem Set 4: possibilities are $y \ln x + x \ln y$ for part (c) and $x\, e^{xy}$ for part (d).

3 (Curl Test II)**.** Let $\mathbf{F}$ be the vector field

$$\mathbf{F}(x,y) = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right) = \frac{1}{r}\,\mathbf{e}_\theta(r,\theta)$$

defined for (x,y) in $\mathbb{R}^2$ with $(x,y) \neq (0,0)$.

(a) Check that $\operatorname{curl}\mathbf{F} = 0$ for all $(x,y) \neq (0,0)$.

(b) Let C be the unit circle centered at the origin, oriented counterclockwise. Check that

$$\int_C \mathbf{F}\cdot d\mathbf{r} = 2\pi.$$

(c) The curl test seems to imply that $\mathbf{F}$ is conservative, as $\operatorname{curl}\mathbf{F} = 0$ at all points where $\mathbf{F}$ is defined by part (a). If $\mathbf{F}$ was conservative, we would have $\int_C \mathbf{F}\cdot d\mathbf{r} = 0$ for every closed curve C. Why doesn't part (b) then contradict the curl test?

Now, let $\mathbf{G}$ be the same vector field, but restricted to the smaller region $Y = \{(x,y) : x > 0\}$.

(d) Check that

$$\mathbf{G} = \nabla\left(\arctan\left(\frac{y}{x}\right)\right).$$

(e) Recall that $\arctan(y/x) = \theta(x,y)$ is the polar angle of the point (x,y). Conclude by the fundamental theorem of calculus for line integrals that for any curve C from point Q to point P in Y,

$$\int_C \mathbf{G}\cdot d\mathbf{r} = \theta(P) - \theta(Q).$$

Remark. For any closed curve, the integral

$$\frac{1}{2\pi}\int_C \mathbf{F}\cdot d\mathbf{r}$$

is called the *winding number* of C about the origin.

4 (Using Green's Theorem to Compute Area)**.** Define the following vector fields on $\mathbb{R}^2$:

$$\mathbf{F}_1(x,y) = \left(-\frac{y}{2}, \frac{x}{2}\right), \qquad \mathbf{F}_2(x,y) = (-y, 0), \qquad \mathbf{F}_3(x,y) = (0, x).$$

Let C be a simple closed curve, and let R be the region bounded by C. Orient C so that R appears on the left as one goes around C.

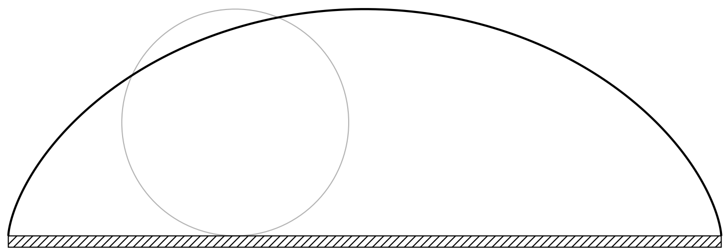
(a) Apply Green's Theorem to show that $\int_C \mathbf{F}_i \cdot d\mathbf{r} = \operatorname{Area}(R)$ for each $i = 1,2,3$.

(b) (Ellipse) Find the area bounded by the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ (try $\mathbf{F}_1$).

(c) (Arc of a Cycloid) Near the beginning of the course, we have seen that the path of a fixed point on the circumference of a unit circle rolling without slipping at unit speed may be parametrized by

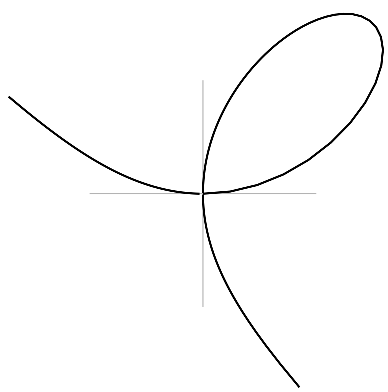
$$t \mapsto (t - \sin(t), 1 - \cos(t)), \quad t \in \mathbb{R}.$$

As t varies in $[0, 2\pi]$, a single arc of the motion is traced out. Let C_1 denote this arc.



The curve C_1 is not closed. However, we can still apply Green's theorem to the piecewise curve $C = C_1 + C_2$, where C_2 is the line segment from $(2\pi, 0)$ to $(0, 0)$! Compute $\int_C \mathbf{F_2} \cdot \mathbf{dr}$, and explain why this is equal to negative of the area under the arc of the cycloid.

(d) (The Folium of Descartes) Find the area of the region bounded by the loop of the folium of Descartes $x^3 + y^3 = 3xy$:



The loop may be parametrized (with orientation as in Green's theorem) by

$$t \mapsto \left(\frac{3t}{1+t^3}, \frac{3t^2}{1+t^3} \right), \quad t \in [0, \infty)$$

(try $\mathbf{F}_3$ — the computation will take a little work).

Remark. The trick used in part (c) — closing up a curve to make it possible to apply Green's theorem — is a useful one.