MTHE 227 Problem Set 4

Due Thursday October 13 2016 at beginning of class

1 (Finding Potentials)**.** For each of the following vector fields $\mathbf{F}$, find a real-valued function f such that $\mathbf{F} = \nabla f$ (problems from *Calculus* by J. Stewart):

(a) $\mathbf{F}(x,y) = \left(\ln y + \frac{y}{x},\ \ln x + \frac{x}{y} \right)$ with $x > 0$, $y > 0$.

(b) $\mathbf{F}(x,y) = \left((1+xy)e^{xy},\ x^2 e^{xy} \right)$ with $(x,y) \in \mathbb{R}^2$.

(c) $\mathbf{F}(x,y,z) = \left(y^2 z + 2xz^2,\ 2xyz,\ xy^2 + 2x^2 z \right)$ with $(x,y,z) \in \mathbb{R}^3$.

2 (Geometric Meaning of Flux)**.** Let C be the unit circle in $\mathbb{R}^2$ with normal pointing away from the origin (note: unlike Problem Set 3, C is the entire circle, not just the upper semicircle). Define the following vector fields:

$$\mathbf{F}(x,y) := (x,y), \qquad \mathbf{G}(x,y) := \left(\frac{x-y}{\sqrt{2}}, \frac{x+y}{\sqrt{2}} \right), \qquad \mathbf{H}(x,y) := (-y,x) \qquad \text{for all } (x,y) \in \mathbb{R}^2$$

(As a reminder, these three vector fields can be obtained by rotating each vector of the field $\mathbf{F}(x,y) = (x,y)$ in place counterclockwise by $0, \pi/4$ and $\pi/2$ radians, respectively.)

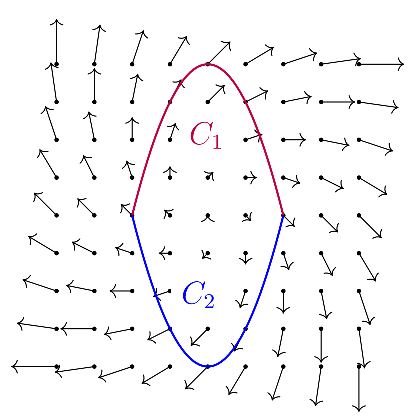
Compute the flux $\int_C \mathbf{F} \cdot \hat{\mathbf{n}}\, ds$, $\int_C \mathbf{G} \cdot \hat{\mathbf{n}}\, ds$, and $\int_C \mathbf{H} \cdot \hat{\mathbf{n}}\, ds$ of each of the three vector fields across C. Which of the three is largest? Which is smallest? Explain briefly. (Be careful to orient the normals correctly.)

Optional Problem. The three vector fields above are members of the family

$$\mathbf{F}_\theta(x,y) = (x\cos\theta - y\sin\theta,\ x\sin\theta + y\cos\theta),$$

with $0 \le \theta < 2\pi$ ($\mathbf{F} = \mathbf{F}_0$, $\mathbf{G} = \mathbf{F}_{\pi/4}$, $\mathbf{H} = \mathbf{F}_{\pi/2}$). The vector field $\mathbf{F}_\theta$ can be obtained by rotating each vector of the field $\mathbf{F}(x,y) = (x,y)$ counterclockwise by θ radians (in place). Plot the flux of $\mathbf{F}_\theta$ across C, as a function of θ.

3 (More Practice with Flux)**.** Let $C = C_1 + C_2$, where C_1 is the graph of the function $x \mapsto 4 - x^2$ with domain $[-2,2]$, and C_2 is the graph of the function $x \mapsto x^2 - 4$ with domain $[-2,2]$. Compute the flux out of the region enclosed by C (this means that the normals are pointing outward) of the vector field $\mathbf{F}(x,y) = (x+y,\ y-x)$.



4. Let C be a simple oriented curve (reminder: this means that C has no self-intersections (except for the possibility that the two endpoints of C may meet), and that one of the two possible orientations of C has been chosen). Denote by $-C$ the simple oriented curve obtained by reversing the orientation of C.

If $t \mapsto \mathbf{r}(t),\ t \in [a,b]$ is a parametrization of C, then one possible parametrization of $-C$ is $t \mapsto \mathbf{r}(b+a-t),\ t \in [a,b]$.

(a) Let L be the line segment in $\mathbb{R}^2$ going from the origin $(0,0)$ to the point $(1,2)$. Parametrize L, denoting your parametrization by $t \mapsto \mathbf{r}(t)$, $t \in [a,b]$. Then, convince yourself that the above prescription produces a parametrization of $-L$: check that $t \mapsto \mathbf{r}(b+a-t),\ t \in [a,b]$ goes from $(1,2)$ to $(0,0)$, and compare its velocity vector with that of your parametrization of L.

(b) Let $\mathbf{F}$ be a vector field. Let C be a simple oriented curve, parametrized as $t \mapsto \mathbf{r}(t)$, with $t \in [a,b]$. Denote the reversed-orientation parametrization above by $t \mapsto \mathbf{s}(t) := \mathbf{r}(b+a-t)$, with $t \in [a,b]$. Show that

$$\int_a^b \mathbf{F}(\mathbf{s}(t)) \cdot \mathbf{s}'(t)\, dt = -\int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt.$$

(*Suggestion:* Expand out in coordinates and make a u-substitution.)

In fact, it is true in general that $\int_{-C} \mathbf{F} \cdot d\mathbf{r} = -\int_C \mathbf{F} \cdot d\mathbf{r}$ (and the proof is similar to the computation above, but involves a discussion of orientation-reversing reparametrizations, which we will omit).

As discussed in lecture, the work integral over a piecewise curve $D = D_1 + D_2 + \cdots + D_n$, where each D_i is simple and oriented, is defined as

$$\int_D \mathbf{F} \cdot d\mathbf{r} := \int_{D_1} \mathbf{F} \cdot d\mathbf{r} + \int_{D_2} \mathbf{F} \cdot d\mathbf{r} + \cdots + \int_{D_n} \mathbf{F} \cdot d\mathbf{r}.$$

For simplicity of notation, write $C + (-C)$ as $C - C$.

(c) Conclude that $\int_{C-C} \mathbf{F} \cdot d\mathbf{r} = 0$.

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Recall the following definition from lecture:

Definition. A vector field $\mathbf{F}$ is called *conservative* if, for any pair of points $Q,\ P$, and any pair of piecewise curves $C,\ C'$ from Q to P, we have $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{C'} \mathbf{F} \cdot d\mathbf{r}$.

Negating the definition, a vector field $\mathbf{F}$ is *not* conservative if there exists *some* pair of points $Q,\ P$, and *some* pair of piecewise curves $C,\ C'$ from Q to P with $\int_C \mathbf{F} \cdot d\mathbf{r} \neq \int_{C'} \mathbf{F} \cdot d\mathbf{r}$.

(d) Show the stronger statement: if a vector field is not conservative, then for *all* pairs of points $Q,\ P$, there exists some pair of piecewise curves $C,\ C'$ from Q to P with $\int_C \mathbf{F} \cdot d\mathbf{r} \neq \int_{C'} \mathbf{F} \cdot d\mathbf{r}$. (*Suggestion:* Extend by two line segments the two piecewise curves guaranteed by the above negation of the definition of conservative.)