# MTHE 227 Problem Set 3

Due Wednesday October 05 2016 at beginning of class

**1.** Let $f$ be the function $f(x, y) = x^2 y + y$. Define the following paths in $\mathbb{R}^2$:

$$\begin{aligned}
&C_1 : \text{The line segment} \quad t \mapsto (0,\, t), \quad t \in [-1,\, 1] \\
&C_2 : \text{The sideways parabola segment} \quad t \mapsto (1 - t^2, t), \quad t \in [-1, 1] \\
&C_3 : \text{The left unit semicircle} \quad t \mapsto (-\cos t,\, \sin t), \quad t \in [-\pi/2, \pi/2].
\end{aligned}$$

Each of the paths connects the points $(0, -1)$ and $(0, 1)$.

(a) Compute $f(0, 1) - f(0, -1)$.

(b) Let $\mathbf{F} = \nabla f$ be the gradient field of $f$. Compute $\mathbf{F}$.

(c) Compute the work $\int_{C_1} \mathbf{F} \cdot \mathbf{dr}$, $\int_{C_2} \mathbf{F} \cdot \mathbf{dr}$, and $\int_{C_3} \mathbf{F} \cdot \mathbf{dr}$ done by $\mathbf{F}$ along each of the $C_i$.

(d) Explain the connection between (a) and (c).

Now define the path

$$C_4 : \text{The line segment} \quad t \mapsto (0,\, -t), \quad t \in [-1,\, 1]$$

(e) Compute $\int_{C_4} \mathbf{F} \cdot \mathbf{dr}$, compare with $\int_{C_1} \mathbf{F} \cdot \mathbf{dr}$, and explain the difference.

**2.** Let $C$ be upper unit semicircle in $\mathbb{R}^2$, oriented clockwise. Define the following vector fields:

$$\mathbf{F}(x, y) := (x, y), \qquad \mathbf{G}(x, y) := \left( \frac{x - y}{\sqrt{2}}, \frac{x + y}{\sqrt{2}} \right), \qquad \mathbf{H}(x, y) := (-y, x) \qquad \text{for all } (x, y) \in \mathbb{R}^2$$

These three vector fields can be obtained by rotating each vector of the field $\mathbf{F}(x, y) = (x, y)$ counterclockwise by $0, \pi/4$ and $\pi/2$ radians, respectively.

Compute $\int_C \mathbf{F} \cdot \mathbf{dr}$, $\int_C \mathbf{G} \cdot \mathbf{dr}$ and $\int_C \mathbf{H} \cdot \mathbf{dr}$. Which of the three is largest? Which is smallest? Explain briefly. (Be careful to parametrize $C$ with the correct orientation.)

*Optional Problem.* The three vector fields above are members of the family

$$\mathbf{F}_\theta(x, y) = (x \cos\theta - y \sin\theta,\, x \sin\theta + y \cos\theta),$$

with $0 \le \theta < 2\pi$ ($\mathbf{F} = \mathbf{F}_0$, $\mathbf{G} = \mathbf{F}_{\pi/4}$, $\mathbf{H} = \mathbf{F}_{\pi/2}$). The vector field $\mathbf{F}_\theta$ can be obtained by rotating each vector of the field $\mathbf{F}(x, y) = (x, y)$ counterclockwise by $\theta$ radians. Plot the work done by $\mathbf{F}_\theta$ along $C$, as a function of $\theta$.

**3** (Potential for a Linear 2D Spring)**.** Suppose we have a spring in $\mathbb{R}^2$ whose length at rest is equal to $\ell$, with one end of the spring attached to a fixed point. Choose a system of coordinates so that the fixed point is at the origin. Suppose that the spring can be rotated freely (without friction) about the pivot at the origin. If the other end of the spring is moved to position $\mathbf{r} \neq \vec{0}$ in $\mathbb{R}^2$, *Hooke's Law* tells us that the force exerted by the spring on a particle (of unit mass) attached to the non-pivoted end is equal to

$$\mathbf{F}(\mathbf{r}) = -k(\|\mathbf{r}\| - \ell)\frac{\mathbf{r}}{\|\mathbf{r}\|}, \tag{1}$$

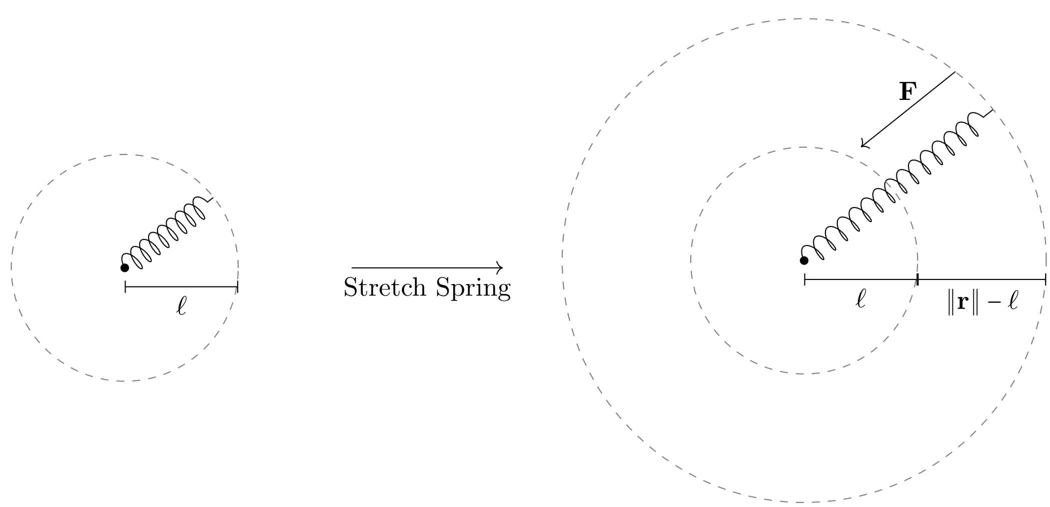
where $k > 0$ is a constant. In other words, the force is proportional to the distance the spring is moved from its resting position, and directed radially toward the resting position.[1] (It is sometimes called a *restoring force.*)

(a) Check that in $(x, y)$-coordinates expression (1) becomes

$$\mathbf{F}(x,y) = \left(-kx + \frac{k\ell x}{\sqrt{x^2+y^2}},\ -ky + \frac{k\ell y}{\sqrt{x^2+y^2}}\right).$$

(b) Find a potential $\varphi$ for this vector field such that $\varphi(\ell, 0) = 0$. Show that the potential can be written as a function of $r := \sqrt{x^2+y^2}$ only. What do the equipotential curves look like?

(c) Find the work required to rotate and stretch the spring from the point $(\ell, 0)$ to the point $(0, R)$, for any $R \geq \ell$.




[1] This is an empirical law that approximates real springs fairly well, as long as $\|\mathbf{r}\|$ is relatively close to $\ell$.