MTHE 227 Problem Set 12

The problem set is due before 10:00 pm on Thursday December 08.

Drop your solutions off in the wooden stacks opposite the lecture hall entrance. I will mark the cell to use for Problem Set 12. In the stacks you can also find previous graded sets you may not have picked up, as well as the midterm (they are in cells marked 'MATH 227').

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1. Let S be the part of the sphere $x^2 + y^2 + z^2 = 4$ contained in the octant $x \geq 0$, $y \geq 0$, $z \geq 0$, oriented outward, and let $C = \partial S$ be the boundary curve of S with the induced orientation (thus, C is a simple closed curve consisting of three arcs).

Let $\mathbf{F}(x, y, z) = (y, -x, z)$. Compute $\iint_S \operatorname{curl} \mathbf{F} \cdot \mathbf{dS}$ in three ways:

(a) Directly compute $\int_C \mathbf{F} \cdot \mathbf{dr}$. (This is equal to $\iint_S \operatorname{curl} \mathbf{F} \cdot \mathbf{dS}$ by Stokes' theorem.)

(b) Directly compute $\iint_S \operatorname{curl} \mathbf{F} \cdot \mathbf{dS}$ by parametrizing S.

(c) Let R be the region enclosed by S and the three coordinate planes. Let D_1, D_2, D_3 be the three quarter-disks making up the boundary of R along with S, each oriented outward (we have $\partial R = S + D_1 + D_2 + D_3$). Explain why

$$\iint_{D_1 + D_2 + D_3} \operatorname{curl} \mathbf{F} \cdot \mathbf{dS} = -\iint_S \operatorname{curl} \mathbf{F} \cdot \mathbf{dS},$$

and directly compute the flux of $\operatorname{curl} \mathbf{F}$ across each of the D_i, hence across S.

2. Let R be the solid cube with vertices at $(\pm 2, \pm 2, \pm 2)$ (so, R has side length 4 and is centered at the origin). Let S be the boundary ∂R of R with the disk $x^2 + y^2 \leq 1, z = 2$ removed from its top face. Let

$$\mathbf{F}(x, y, z) = (xy^2 + \arctan(y^2 z),\ y^3 - e^{-x(z-2)^2},\ \cos(\pi z)).$$

(a) Apply the divergence theorem to compute the flux of $\mathbf{F}$ through ∂R.

(b) Directly compute the flux of $\mathbf{F}$ through the disk $x^2 + y^2 \leq 1, z = 2$, with the normal pointing up (that is, along the positive z-direction).

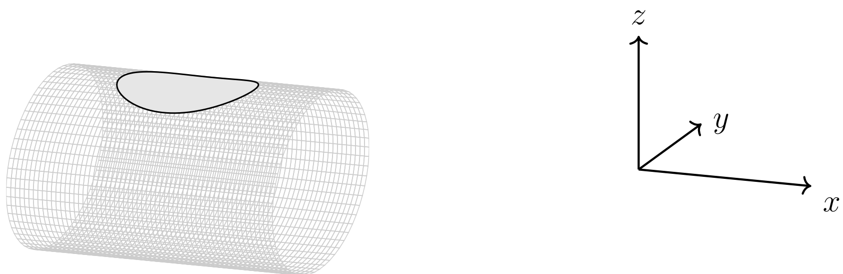
(c) Combine the results of parts (a) and (b) to compute the flux through S, with the normal pointing outward.

3. (a) The intersection of the (hollow) cylinders $x^2 + y^2 = 1$ and $y^2 + z^2 = 4$ consists of two pieces, each of which is a simple closed curve. Parametrize the piece with $z \geq 0$. (Suggestion: First parametrize its shadow in the xy-plane.)

(b) If $\mathbf{G}(x, y, z) = (x^2 z,\ y^2 x,\ -z^2 x)$ and $\mathbf{F}(x, y, z) = (0,\ x^2 + z^2,\ y^2)$, check that $\operatorname{curl} \mathbf{G} = \mathbf{F}$.

(c) Let S be the surface described by $x^2 + y^2 \leq 1,\ y^2 + z^2 = 4,\ z \geq 0$. Compute the flux $\iint_S \mathbf{F}\cdot\mathbf{dS}$, with normals pointing out of the cylinder $y^2+z^2 = 4$ (that is, with $\mathbf{N}\cdot\mathbf{e_z} \geq 0$).

Possibly useful identities: $\cos^2(x) = \dfrac{1+\cos(2x)}{2}, \quad \sin^2(x) = \dfrac{1-\cos(2x)}{2}$.



4. If S is a closed surface bounding a region R, and $\mathbf{F}$ is a vector field defined everywhere in R, then by the divergence theorem we have

$$\iint_S \operatorname{curl}\mathbf{F}\cdot\mathbf{dS} = \iiint_R \operatorname{div}(\operatorname{curl}\mathbf{F})\,dV = 0,$$

since $\operatorname{div}(\operatorname{curl}\mathbf{F}) = 0$ identically. This is analogous to the fact that $\int_C \nabla f\cdot\mathbf{dr} = 0$ for a closed curve.

Let $\mathbf{F}(x,y,z) = \left(\dfrac{x}{(x^2+y^2+z^2)^{3/2}}, \dfrac{y}{(x^2+y^2+z^2)^{3/2}}, \dfrac{z}{(x^2+y^2+z^2)^{3/2}}\right)$ with $(x,y,z) \neq (0,0,0)$. Check that $\operatorname{div}\mathbf{F} = 0$ everywhere $\mathbf{F}$ is defined, but that $\iint_S \mathbf{F}\cdot\mathbf{dS} = 4\pi$, where S is the unit sphere $x^2+y^2+z^2 = 1$.

Optional Problem. Define a 'toric change of variables' $T\colon D \to \mathbb{R}^3_{(x,y,z)}$, where

$$D = \{(r,\theta,t) \in \mathbb{R}^3 : 0 \leq r < b,\ 0 \leq t \leq 2\pi,\ 0 \leq \theta \leq 2\pi\},$$

by

$$\begin{aligned} x(r,\theta,t) &= (b + r\cos(t))\cos(\theta),\\ y(r,\theta,t) &= (b + r\cos(t))\sin(\theta),\\ z(r,\theta,t) &= r\sin(t). \end{aligned}$$

(a) Find a region V^* in $\mathbb{R}^3_{(r,\theta,t)}$ such that its image $V = T(V^*)$ is the region bounded by the torus with radii a and b.

(b) Check that $\det \dfrac{\partial(x,y,z)}{\partial(r,\theta,t)} = r(b + r\cos(t))$.

(c) Apply the change of variables theorem to conclude that the integral

$$\iiint_{V^*} r(b + r\cos(t))\,dt\,dr\,d\theta$$

is equal to the volume of V, and compute the integral.