

MTHE 227 PROBLEM SET 10
Due Thursday November 24 2016 at the beginning of class

1. (a) Sketch the cross-section of the (hollow) cylinder $y^2 + z^2 = 4$ in the xz -plane, as well as the vector field

$$\mathbf{F}(x, y, z) = \begin{cases} \left(1 - \frac{y^2+z^2}{4}, 0, 0\right), & y^2 + z^2 < 4 \\ \mathbf{0}, & \text{Otherwise} \end{cases}.$$

in this cross-section.

This is a simple model of water flowing through a pipe without turbulence (interestingly, the velocity goes to zero at the boundary!).

- (b) The disk S described by $x = x_0$, $y^2 + z^2 \leq 4$ (the region bounded by the cross-section of the pipe in the plane $x = x_0$) may be parametrized by

$$(u, v) \mapsto (x_0, u \cos(v), u \sin(v)), \quad u \in [0, 2], v \in [0, 2\pi].$$

Find the flux $\iint_S \mathbf{F} \cdot d\mathbf{S}$ of the vector field \mathbf{F} of part (a) through S , with the normal pointing along \mathbf{e}_x (that is, along the positive x direction).

- (c) The portion Σ of the cylinder $y^2 + z^2 = 1$ between $x = 1$ and $x = 2$ may be parametrized by

$$(u, v) \mapsto (v, \cos(u), \sin(u)), \quad u \in [0, 2\pi], v \in [1, 2].$$

The surface Σ has the same central axis as the cylinder $y^2 + z^2 = 4$, and is contained within the cylinder. Check that $\iint_{\Sigma} \mathbf{F} \cdot d\mathbf{S} = 0$, and briefly explain.

2. Sketch the volume of integration for the iterated integral $\int_0^1 \int_0^y \int_0^x x^2 y z \, dz \, dx \, dy$, and express it in the five other possible orders of integration.

(You do not have to evaluate any of the integrals.)

3. Describe the volume of integration, convert to cylindrical coordinates, and evaluate

(a) $\int_{-3}^3 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_{\sqrt{x^2+y^2}}^3 dz \, dy \, dx$,

(b) $\int_{-\sqrt{8}}^{\sqrt{8}} \int_{-\sqrt{8-x^2}}^{\sqrt{8-x^2}} \int_{x^2+y^2-8}^{8-x^2-y^2} 2z \, dz \, dy \, dx$.

4. (a) Show that the center of mass of the solid $x^2 + y^2 + z^2 \leq 1$, $z \geq 0$ (the top half of the ball of radius 1) of uniform density δ has Cartesian coordinates $(0, 0, \frac{3}{8})$. (Suggestion: Integrate in spherical coordinates.)

(b) Now suppose that the density δ of the solid in part (a) is given by

$$\delta(x, y, z) = 1 - \gamma z$$

for some number $0 \leq \gamma \leq 1$. (Interpretation: the ball is made of lighter material at the top than at the base. The upper bound on γ is made to avoid regions of negative density.)

Find the coordinates of the center of mass as a function of γ . For which γ is the center of mass at the point $(0, 0, \frac{1}{3})$?