

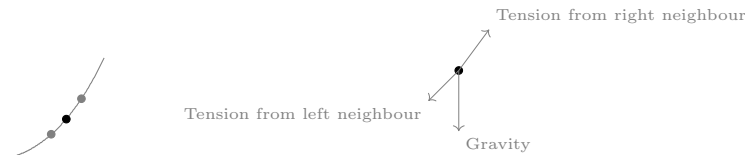
THE CATENARY
A Supplemental Note

Problem. Describe the shape of a wire of uniform density ρ hanging in uniform gravity from two supports of equal height.

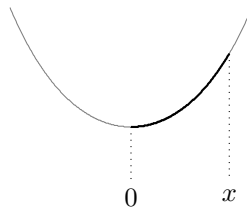
We solve the problem by finding a function f whose graph describes the shape of the wire.

Setting up. Choose a frame of reference so that the lowest point of the wire is at $x = 0$. Notice that this choice of frame of reference implies that $f'(0) = 0$. (The y -coordinate of the origin of this frame of reference is left arbitrary through the argument.)

Think of the wire as composed of a number of point particles arranged along a curve, and the tension in the wire as caused by forces between neighbouring particles. Each individual particle has three forces acting on it: the force of gravity, and the two pulls of tension due to its neighbouring particles (we are assuming that the pulls due to the rest of the particles are effectively zero, because of their further distance). When the number of particles is very large, the two tension forces have directions that are nearly tangent to the wire.



Now mentally isolate a piece of the wire between $x = 0$ and some $x > 0$.



The forces of the particles within this piece of wire on each other cancel in pairs by Newton's Third Law, so we can think of the entire piece as pulled by the force of tension \vec{T} caused by the 'next particle to the right' (or else the right support), as well as the force of tension \vec{T}_0 caused by the 'next particle to the left'. From our picture with interacting particles, we see that the force \vec{T} is directed along the tangent to the graph of f at x , and the force \vec{T}_0 is directed along the tangent to the graph of f at 0; the force \vec{T}_0 is therefore horizontal.¹

There is also the force of gravity acting on the piece of wire, which is directed down and has magnitude equal to mass of the piece of wire times the gravitational constant g . Finding the mass of the piece of wire is a problem we know how to solve — it is precisely equal to the integral

$$\int_0^x \rho \sqrt{1 + f'(t)^2} dt.$$

¹Similar arguments would show that any isolated piece of the wire is pulled by two tension forces tangent to the wire at the endpoints of the isolated piece. The choice of the bottom point is a good one, because the tangent line at the bottom point is horizontal.

Indeed, the wire can be parametrized by $t \mapsto (t, f(t))$, with velocity $(1, f'(t))$ and speed $\sqrt{1 + f'(t)^2}$, so the above is precisely the expression for the integral of ρ over the isolated piece of wire.

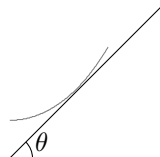
Let θ denote the angle that the force \vec{T} makes with the positive x -axis. Because the wire hangs in equilibrium, the vertical and horizontal components of the forces acting on the isolated piece of wire cancel, and we find that

$$\begin{aligned} \|\vec{T}\| \cos \theta &= \|\vec{T}_0\|, \\ \|\vec{T}\| \sin \theta &= \underbrace{\left(\int_0^x \rho \sqrt{1 + f'(t)^2} dt \right)}_{\text{mass of piece of wire}} g. \end{aligned}$$

Dividing the second expression by the first, we find

$$\tan \theta = \frac{\rho g}{\|\vec{T}_0\|} \int_0^x \sqrt{1 + f'(t)^2} dt.$$

As discussed above, the force \vec{T} is tangent to the graph of f at x . This implies that the force \vec{T} makes the same angle with the positive x -axis as does the tangent line to the graph of f at x . We called this angle θ above. Looking at the picture below, we see that $\tan \theta$ is equal to the slope of the tangent line to f at x , which in turn is equal to $f'(x)$ by definition.



$$\tan \theta = \frac{\text{rise of tangent line}}{\text{run of tangent line}} = \text{slope of tangent line}.$$

We thus have

$$f'(x) = \frac{\rho g}{\|\vec{T}_0\|} \int_0^x \sqrt{1 + f'(t)^2} dt.$$

Finally, recall the following theorem: ²

Theorem (Fundamental Theorem of Calculus). *Let $f(x)$ be a continuous function on an interval $[a, b]$, and define a function $F(x)$ by*

$$F(x) = \int_a^x f(t) dt.$$

Then F is differentiable on (a, b) , and $F'(x) = f(x)$.

Applying the theorem, differentiating the equation found above with respect to x , we find the following differential equation (you may object that we assumed $x > 0$ above, but this was purely for simplicity of exposition — completely symmetric arguments may be made for other x):

$$f''(x) = \frac{\rho g}{\|\vec{T}_0\|} \sqrt{1 + f'(x)^2}.$$

²As you may know, the Fundamental Theorem of Calculus has a quite intuitive interpretation, which however this is not the place to discuss (I would be happy to talk about it elsewhere however, for instance in office hours!).

Reminder of the Analytic Definitions of cosh and sinh. These are

$$\cosh(x) = \frac{e^x + e^{-x}}{2} \quad \text{and} \quad \sinh(x) = \frac{e^x - e^{-x}}{2}.$$

It is useful to know that

$$\frac{d}{dx} \cosh(x) = \sinh(x), \quad \frac{d}{dx} \sinh(x) = \cosh(x), \quad \text{and} \quad \cosh^2(x) - \sinh^2(x) = 1,$$

all of which you can check quite readily.

Solving the Equation. To make the notation cleaner, let's write $a = \frac{\rho g}{\|T_0\|}$ and $z(x) = f'(x)$ (the appearance of the letter z is not intended to point out a connection with complex numbers, if there is any!). The differential equation we are to solve is then

$$\frac{dz}{dx} = a\sqrt{1+z^2}.$$

We are in luck, because this equation is *separable*. Equations of this type can be solved by an algorithm, which, if it was not already, will certainly be covered in your ODEs course. Following this algorithm:

$$\text{separate the variables} \quad \frac{dz}{\sqrt{1+z^2}} = a \, dx \quad \text{and integrate} \quad \int \frac{dz}{\sqrt{1+z^2}} = \int a \, dx.$$

To find the left integral, make the following substitution: $z = \sinh(u)$, $dz = \cosh(u) \, du$. This is useful, because from

$$\cosh^2(u) - \sinh^2(u) = 1,$$

we find

$$\cosh^2(u) = 1 + \sinh^2(u) \quad \text{and therefore} \quad \cosh(u) = \sqrt{1 + \sinh^2(u)}.$$

The left integral is therefore

$$\int \frac{dz}{\sqrt{1+z^2}} = \int \frac{\cosh(u) \, du}{\cosh(u)} = \int du = u = \sinh^{-1}(z).$$

The right integral is simple:

$$\int a \, dx = ax + C.$$

Hence, we find that

$$z(x) = \sinh(ax + C)$$

is a general solution. Now, remembering that $z(x) = f'(x)$ by definition, and that we chose our frame of reference so that the derivative $f'(0) = 0$ (the lowest point of the wire is at $x = 0$), we need

$$0 = f'(0) = \sinh(C) = \frac{e^C - e^{-C}}{2},$$

which implies that $e^C = e^{-C}$. The unique solution to the last equation is $C = 0$ (if you are not convinced, you can verify this by taking \ln of both sides). Therefore,

$$f'(x) = \sinh(ax).$$

Now, integrate $f'(x)$ to find

$$f(x) = \frac{1}{a} \cosh(ax) + D.$$

What is the interpretation of the constant D ? Let h denote the height of the lowest point of the wire above our choice of reference point. Remembering that the lowest point of the wire is at $x = 0$ by our choice of coordinates,

$$h = f(0) = \frac{1}{a} \cosh(0) + D = \frac{1}{a} + D.$$

So D is related to the height of the lowest point.

Conclusion. *The wire is described by the graph of the function*

$$f(x) = \frac{\|\vec{T}_0\|}{\rho g} \left(\cosh\left(\frac{\rho g}{\|\vec{T}_0\|} x\right) - 1 \right) + h,$$

where ρ is the density of the wire, g is the gravitational constant, $\|\vec{T}_0\|$ is the magnitude of the force of tension at the lowest point of the wire, and h is the height of the lowest point above the reference point. Such a curve is called a catenary.

It is convenient to choose the y -coordinate of the frame of reference to lie $\|\vec{T}_0\|/(\rho g)$ below the bottom point of the catenary. With this choice, the function describing the catenary takes the simpler form

$$f(x) = \frac{\|\vec{T}_0\|}{\rho g} \cosh\left(\frac{\rho g}{\|\vec{T}_0\|} x\right) = \frac{1}{a} \cosh(ax).$$

Exercise. (a) *Find the arclength of the catenary $x \mapsto (x, \cosh(x))$ between $-L$ and L . Do the same for a general catenary $x \mapsto (x, \cosh(ax)/a)$, $a > 0$.*

Answers. $2 \sinh(L) = e^L + e^{-L}$; $2 \sinh(aL)/a$.

(b) *Compare the next few hanging cables, chains and wires you see with the graph below!*

