Mthe 237 — Problem Set 10 Solutions

1. For each of the following systems, sketch the flow lines of the system, and find a parametrization of the flow line that passes through the given point.

i) $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \qquad x(0) = 4,\ y(0) = 0.$

ii) $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 4 & -1 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \qquad x(0) = 1,\ y(0) = 0.$

iii) $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -1 & 5 \\ 5 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \qquad x(0) = 0,\ y(0) = 2.$

Solution. i) Let's begin by finding the eigenvalues and eigenvectors of

$$A = \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}$$

(these were found in Homework 9 as well, so one could use the answer found there).

The characteristic polynomial of A is

$$p_A(z) = \det(A - zI) = \det \begin{pmatrix} 2-z & 3 \\ 1 & 4-z \end{pmatrix} = (2-z)(4-z) - 3 = z^2 - 6z + 5 = (z-1)(z-5).$$

So the eigenvalues of A are $\lambda_1 = 1$ and $\lambda_2 = 5$. Since A has distinct eigenvalues, it is diagonalizable (each eigenvalue has at least one corresponding eigenvector).

Let's now find the eigenvectors of A corresponding to these two eigenvalues.

To find an eigenvector corresponding to $\lambda_1 = 1$, look in

$$\ker \begin{pmatrix} 2-1 & 3 \\ 1 & 4-1 \end{pmatrix} = \ker \begin{pmatrix} 1 & 3 \\ 1 & 3 \end{pmatrix}.$$

The condition for $\begin{pmatrix} x \\ y \end{pmatrix}$ to be an eigenvector is that $x + 3y = 0$ (the second row is redundant, as it should be), so

$$\mathbf{v_1} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$$

is an eigenvector of A corresponding to $\lambda_1 = 1$.

To find an eigenvector corresponding to $\lambda_2 = 5$, look in

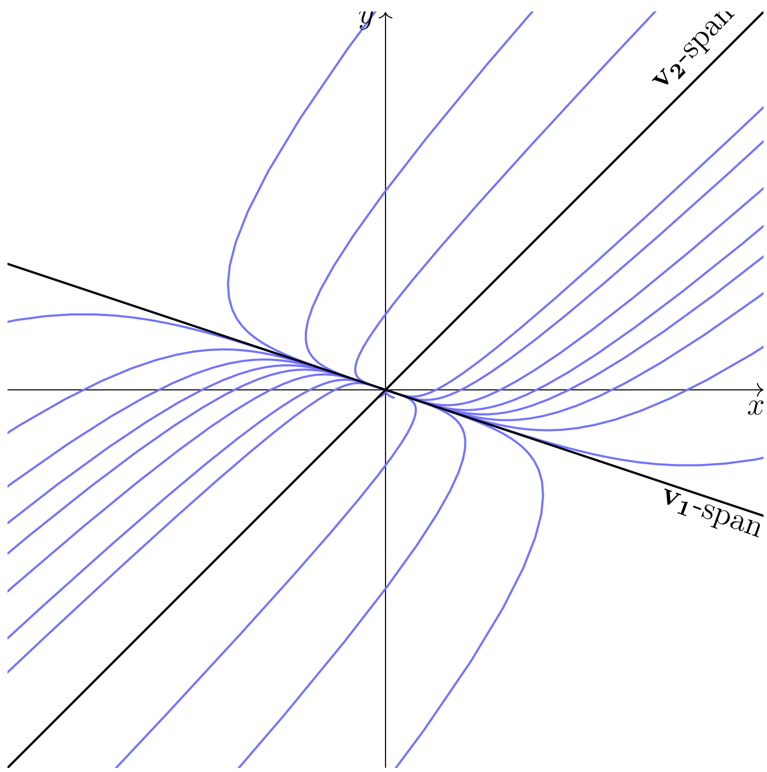
$$\ker \begin{pmatrix} 2-5 & 3 \\ 1 & 4-5 \end{pmatrix} = \ker \begin{pmatrix} -3 & 3 \\ 1 & -1 \end{pmatrix}.$$

The condition for $\begin{pmatrix} x \\ y \end{pmatrix}$ to be an eigenvector is that $-3x + 3y = 0$, so

$$\mathbf{v_2} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

is an eigenvector of A corresponding to $\lambda_2 = 5$.

We have $0 < \lambda_1 < \lambda_2$, so the flow lines describe an unstable node (also called a source), with the flow lines tending to be parallel with the $\mathbf{v_1}$ direction as $t \to -\infty$ and parallel to the $\mathbf{v_2}$ direction as $t \to \infty$.



Now, to find the flow line through a particular point, we compute the matrix exponential $\exp(At)$. By the procedure worked out in lecture, we have

$$\exp(At) = Q \begin{pmatrix} e^t & 0 \\ 0 & e^{5t} \end{pmatrix} Q^{-1},$$

where $Q = \begin{pmatrix} 3 & 1 \\ -1 & 1 \end{pmatrix}$ is the change-of-basis matrix from the eigenbasis to the ‘initial’ basis. Computing,

$$\begin{aligned}
\exp(At) &= \begin{pmatrix} 3 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} e^t & 0 \\ 0 & e^{5t} \end{pmatrix} \left(\frac{1}{4} \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix} \right) \\
&= \frac{1}{4} \begin{pmatrix} 3 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} e^t & -e^t \\ e^{5t} & 3e^{5t} \end{pmatrix} \\
&= \frac{1}{4} \begin{pmatrix} 3e^t + e^{5t} & -3e^t + 3e^{5t} \\ -e^t + e^{5t} & e^t + 3e^{5t} \end{pmatrix}
\end{aligned}$$

By a theorem in class, the solution (or flow line) of $\dot{\mathbf{x}} = A\mathbf{x}$ with initial condition $\mathbf{x}(t_0) = \mathbf{x}_0$ is

$$\mathbf{x}(t) = \exp(A(t - t_0))\mathbf{x}_0.$$

In this problem, $t_0 = 0$, $\mathbf{x}_0 = \begin{pmatrix} 4 \\ 0 \end{pmatrix}$ and

$$\mathbf{x}(t) = \frac{1}{4} \begin{pmatrix} 3e^t + e^{5t} & -3e^t + 3e^{5t} \\ -e^t + e^{5t} & e^t + 3e^{5t} \end{pmatrix} \begin{pmatrix} 4 \\ 0 \end{pmatrix} = \begin{pmatrix} 3e^t + e^{5t} \\ -e^t + e^{5t} \end{pmatrix}.$$

ii) Begin by finding the eigenvalues and eigenvectors of

$$B = \begin{pmatrix} 4 & -1 \\ 3 & 2 \end{pmatrix}.$$

The characteristic polynomial is

$$\begin{aligned}
p_B(z) &= \det(B - zI) \\
&= \det \begin{pmatrix} 4 - z & -1 \\ 3 & 2 - z \end{pmatrix} \\
&= (4 - z)(2 - z) + 3 \\
&= 8 - 6z + z^2 + 3 \\
&= z^2 - 6z + 11.
\end{aligned}$$

The roots of p_B can be found using the quadratic formula —

$$\frac{6 \pm \sqrt{6^2 - 44}}{2} = \frac{6 \pm \sqrt{-8}}{2} = 3 \pm i\sqrt{2}.$$

Let's find an eigenvector for $\lambda_1 = 3 + i\sqrt{2}$.

We have

$$\ker \begin{pmatrix} 4 - (3 + i\sqrt{2}) & -1 \\ 3 & 2 - (3 + i\sqrt{2}) \end{pmatrix} = \ker \begin{pmatrix} 1 - i\sqrt{2} & -1 \\ 3 & -1 - i\sqrt{2} \end{pmatrix}.$$

So,

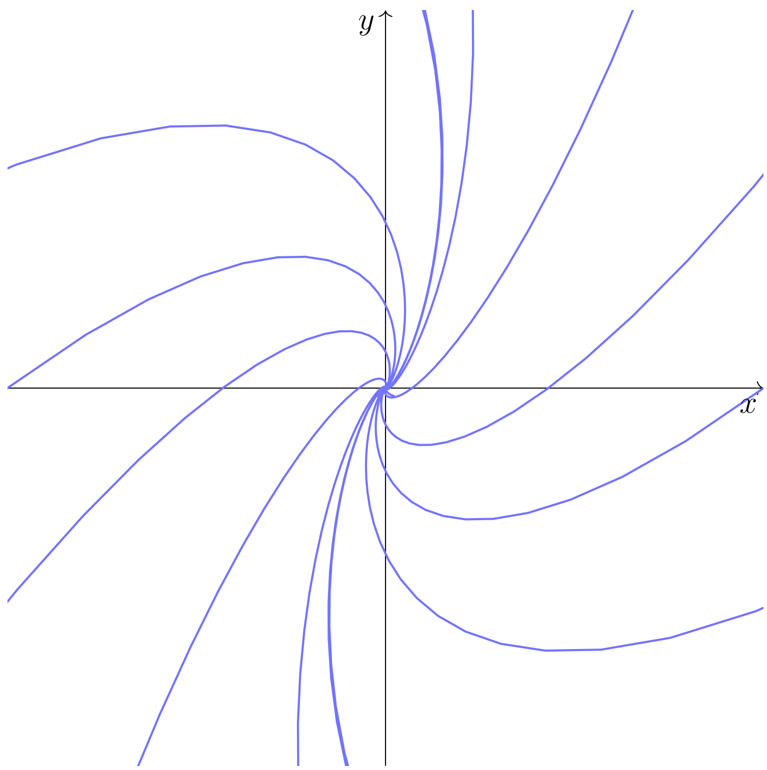
$$\mathbf{v_1} = \begin{pmatrix} 1 \\ 1 - i\sqrt{2} \end{pmatrix}$$

is an eigenvector of B with eigenvalue $3 + i\sqrt{2}$.

Taking the real and imaginary parts of $\mathbf{v_1}$, we get

$$\mathbf{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad \mathbf{y} = \begin{pmatrix} 0 \\ -\sqrt{2} \end{pmatrix}.$$

The flow lines are unstable spirals (unstable because the real part of the eigenvalue is greater than zero).



(The spirals grow too quickly to show several rotations in the above picture.)

The matrix exponential is

$$\begin{aligned}
\exp(Bt) &= \begin{pmatrix} 1 & 0 \\ 1 & -\sqrt{2} \end{pmatrix} \left(e^{3t} \begin{pmatrix} \cos(\sqrt{2}t) & \sin(\sqrt{2}t) \\ -\sin(\sqrt{2}t) & \cos(\sqrt{2}t) \end{pmatrix} \right) \frac{1}{-\sqrt{2}} \begin{pmatrix} -\sqrt{2} & 0 \\ -1 & 1 \end{pmatrix} \\
&= \frac{e^{3t}}{-\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 1 & -\sqrt{2} \end{pmatrix} \begin{pmatrix} -\sqrt{2}\cos(\sqrt{2}t) - \sin(\sqrt{2}t) & \sin(\sqrt{2}t) \\ \sqrt{2}\sin(\sqrt{2}t) - \cos(\sqrt{2}t) & \cos(\sqrt{2}t) \end{pmatrix} \\
&= \frac{e^{3t}}{-\sqrt{2}} \begin{pmatrix} -\sqrt{2}\cos(\sqrt{2}t) - \sin(\sqrt{2}t) & \sin(\sqrt{2}t) \\ -3\sin(\sqrt{2}t) & \sin(\sqrt{2}t) - \sqrt{2}\cos(\sqrt{2}t) \end{pmatrix} \\
&= e^{3t} \begin{pmatrix} \cos(\sqrt{2}t) + \frac{\sin(\sqrt{2}t)}{\sqrt{2}} & -\frac{\sin(\sqrt{2}t)}{\sqrt{2}} \\ \frac{3\sin(\sqrt{2}t)}{\sqrt{2}} & \cos(\sqrt{2}t) - \frac{\sin(\sqrt{2}t)}{\sqrt{2}} \end{pmatrix}
\end{aligned}$$

Therefore, the solution passing through $\begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is

$$\mathbf{x}(t) = e^{3t} \begin{pmatrix} \cos(\sqrt{2}t) + \frac{\sin(\sqrt{2}t)}{\sqrt{2}} \\ \frac{\sin(\sqrt{2}t)}{\sqrt{2}} \end{pmatrix}.$$

iii) Once again, we begin by finding the eigenvalues and eigenvectors of

$$C = \begin{pmatrix} -1 & 5 \\ 5 & -1 \end{pmatrix}.$$

The characteristic polynomial is

$$p_C(z) = \det(C - zI) = \det \begin{pmatrix} -1-z & 5 \\ 5 & -1-z \end{pmatrix} = (-1-z)^2 - 25 = (z+1)^2 - 25 = (z-4)(z+6).$$

So the eigenvalues of C are $\lambda_1 = 4$ and $\lambda_2 = -6$. Once again, since A has distinct eigenvalues, it is diagonalizable.

Let's find the corresponding eigenvectors.

The eigenvectors corresponding to $\lambda_1 = 4$ lie in

$$\ker \begin{pmatrix} -1-4 & 5 \\ 5 & -1-4 \end{pmatrix} = \ker \begin{pmatrix} -5 & 5 \\ 5 & -5 \end{pmatrix}.$$

So the condition for $\begin{pmatrix} x \\ y \end{pmatrix}$ to be an eigenvector is $-5x + 5y = 0$.

We see that

$$\mathbf{v_1} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

is an eigenvector corresponding to $\lambda_1 = 4$.

The eigenvectors corresponding to $\lambda_2 = -6$ lie in

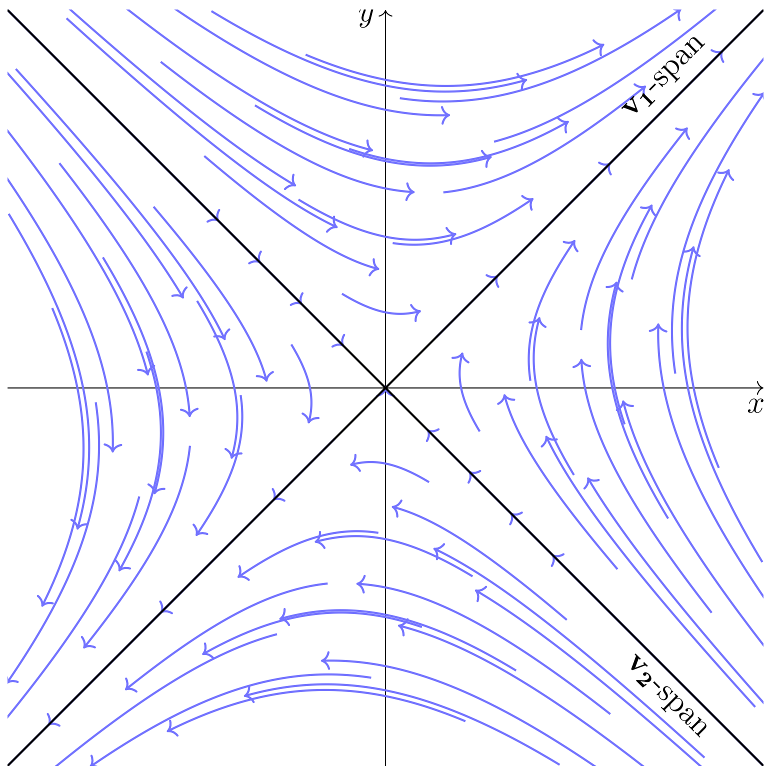
$$\ker \begin{pmatrix} -1+6 & 5 \\ 5 & -1+6 \end{pmatrix} = \ker \begin{pmatrix} 5 & 5 \\ 5 & 5 \end{pmatrix}.$$

The condition is $5x + 5y = 0$, so

$$\mathbf{v_2} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

is an eigenvector corresponding to $\lambda_2 = -6$.

We have $\lambda_2 < 0 < \lambda_1$, so the flow lines describe a saddle.



Finally, we compute the matrix exponential.

We have

$$\begin{aligned}\exp(At) &= \begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}e^{4t} & 0\\ 0 & e^{-6t}\end{pmatrix}\frac{1}{-2}\begin{pmatrix}-1 & -1\\ -1 & 1\end{pmatrix}\\ &= \frac{1}{-2}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}-e^{4t} & -e^{4t}\\ -e^{-6t} & e^{-6t}\end{pmatrix}\\ &= \frac{1}{-2}\begin{pmatrix}-e^{4t}-e^{-6t} & -e^{4t}+e^{-6t}\\ -e^{4t}+e^{-6t} & -e^{4t}-e^{-6t}\end{pmatrix}\\ &= \frac{1}{2}\begin{pmatrix}e^{4t}+e^{-6t} & e^{4t}-e^{-6t}\\ e^{4t}-e^{-6t} & e^{4t}+e^{-6t}\end{pmatrix}.\end{aligned}$$

It follows that the flow line satisfying $\mathbf{x}(0) = \begin{pmatrix}0\\ 2\end{pmatrix}$ is

$$\frac{1}{2}\begin{pmatrix}e^{4t}+e^{-6t} & e^{4t}-e^{-6t}\\ e^{4t}-e^{-6t} & e^{4t}+e^{-6t}\end{pmatrix}\begin{pmatrix}0\\ 2\end{pmatrix} = \begin{pmatrix}e^{4t}-e^{-6t}\\ e^{4t}+e^{-6t}\end{pmatrix}.$$

2. Let

$$A = \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix}1 & 0\\ -1 & 1\end{pmatrix}.$$

Show that $AB \neq BA$ and $\exp(A+B) \neq \exp(A)\exp(B)$.

Solution. We have

$$AB = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1-1 & 0+1 \\ 0-1 & 0+1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$$

and

$$BA = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1+0 & 1+0 \\ -1+0 & -1+1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}.$$

So $AB \neq BA$.

To compute the exponentials $\exp(A)$ and $\exp(B)$, we use the property proved in class that if $CD = DC$, then $\exp(C+D) = \exp(C)\exp(D)$. We have

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

so

$$\exp(A) = \exp\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) \exp\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right).$$

Since

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

we have

$$\exp\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right) = I + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \frac{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}^2}{2!} + \frac{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}^3}{3!} + \cdots = I + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Therefore,

$$\exp\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) \exp\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} e & 0 \\ 0 & e \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} e & e \\ 0 & e \end{pmatrix}.$$

Similarly,

$$B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}.$$

So,

$$\exp(B) = \exp\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) \exp\left(\begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}\right).$$

Since

$$\begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

we have

$$\exp\left(\begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}\right) = I + \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} + 0 = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}.$$

Therefore,

$$\exp\left(\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right)\exp\left(\begin{pmatrix}0 & 0\\ -1 & 0\end{pmatrix}\right) = \begin{pmatrix}e & 0\\ 0 & e\end{pmatrix}\begin{pmatrix}1 & 0\\ -1 & 1\end{pmatrix} = \begin{pmatrix}e & 0\\ -e & e\end{pmatrix}.$$

Finally,

$$\exp(A)\exp(B) = \begin{pmatrix}e & e\\ 0 & e\end{pmatrix}\begin{pmatrix}e & 0\\ -e & e\end{pmatrix} = \begin{pmatrix}e^2 - e^2 & 0 + e^2\\ 0 - e^2 & 0 + e^2\end{pmatrix} = \begin{pmatrix}0 & e^2\\ -e^2 & e^2\end{pmatrix}.$$

On the other hand,

$$A + B = \begin{pmatrix}2 & 1\\ -1 & 2\end{pmatrix}.$$

This is of the form

$$\begin{pmatrix}\sigma & \omega\\ -\omega & \sigma\end{pmatrix}$$

and we know that

$$\exp\left(\begin{pmatrix}\sigma & \omega\\ -\omega & \sigma\end{pmatrix}\right) = \begin{pmatrix}e^\sigma\cos(\omega) & e^\sigma\sin(\omega)\\ -e^\sigma\sin(\omega) & e^\sigma\cos(\omega)\end{pmatrix}.$$

Therefore,

$$\exp(A + B) = \begin{pmatrix}e^2\cos(1) & e^2\sin(1)\\ -e^2\sin(1) & e^2\cos(1)\end{pmatrix}.$$

So,

$$\exp(A + B) \neq \exp(A)\exp(B).$$

(For instance, the top left entry of $\exp(A + B)$ is $e^2\cos(1) \neq 0$ and the top left entry of $\exp(A)\exp(B)$ is 0.)

3. Let $0 < \lambda_1 < \lambda_2$. Show that for the solution of the system

$$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}, \qquad x(0) = x_0 \neq 0,\ y(0) = y_0,$$

the slopes of the tangent lines to the solution tend toward 0 as the solution approaches the origin (that is, as $t \to -\infty$).

(*Suggestion:* One way to approach this is to make use of the identity

$$\dot{y}(t_0)/\dot{x}(t_0) = \text{Slope of tangent line to } t \mapsto (x(t), y(t)) \text{ at } (x(t_0), y(t_0)).)$$

Solution. Let $A = \begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}$. Because A is already diagonal, its matrix exponential is simple to compute —

$$\exp(At) = \begin{pmatrix}e^{\lambda_1 t} & 0\\ 0 & e^{\lambda_2 t}\end{pmatrix}.$$

Therefore, the solution with $\begin{pmatrix}x(0)\\ y(0)\end{pmatrix} = \begin{pmatrix}x_0\\ y_0\end{pmatrix}$ is

$$t \mapsto \exp(At)\begin{pmatrix}x_0\\ y_0\end{pmatrix} = \begin{pmatrix}x_0 e^{\lambda_1 t}\\ y_0 e^{\lambda_2 t}\end{pmatrix}.$$

Notice that indeed $x(t) = x_0 e^{\lambda_1 t} \to 0$ as $t \to -\infty$ and $y(t) = y_0 e^{\lambda_2 t} \to 0$ as $t \to -\infty$, because both λ_1 and λ_2 are positive.

By the hint, the slope of the tangent line to the solution at $(x(t), y(t))$ is

$$\frac{\dot{y}(t)}{\dot{x}(t)} = \frac{\lambda_2 y_0 e^{\lambda_2 t}}{\lambda_1 x_0 e^{\lambda_1 t}} = \frac{\lambda_2 y_0}{\lambda_1 x_0} e^{(\lambda_2 - \lambda_1)t}.$$

Because $\lambda_2 > \lambda_1$, the exponent is positive, and it follows that

$$\lim_{t\to-\infty} \frac{\dot{y}(t)}{\dot{x}(t)} = \lim_{t\to-\infty} \frac{\lambda_2 y_0}{\lambda_1 x_0} e^{(\lambda_2 - \lambda_1)t} = 0,$$

which is what we wanted to show.

Alternative solution. Suppose that the solutions lie on graphs of functions $y(x)$ (this is the case for every solution except the ones with $x(0) = 0$, which are excluded by the hypotheses). From the system of differential equations

$$\begin{aligned} \dot{x} &= \lambda_1 x \\ \dot{y} &= \lambda_2 y, \end{aligned}$$

we have

$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{\lambda_2 y}{\lambda_1 x}.$$

This is a separable equation.

$$\frac{1}{y}\frac{dy}{dx} = \frac{\lambda_2}{\lambda_1}\frac{1}{x}.$$

Integrating both sides,

$$\ln|y| = \ln\left|x^{\lambda_2/\lambda_1}\right| + C$$

so

$$y(x) = Cx^{\lambda_2/\lambda_1}.$$

The initial condition is $y(x_0) = y_0$. Using this, we have $C = y_0/x_0^{\lambda_2/\lambda_1}$. However, we keep writing C for simplicity. Notice that indeed $\lim_{x\to 0} y(x) = 0$.

Now, the tangent line to the graph of $x \to y(x)$ at (x_0, y_0) has slope $y'(x_0)$. Differentiating,

$$y'(x) = C\frac{\lambda_2}{\lambda_1} x^{(\lambda_2-\lambda_1)/\lambda_1}.$$

Because $\lambda_2 > \lambda_1$, the exponent of x is still positive, and we have

$$\lim_{x\to 0} y'(x) = \lim_{x\to 0} C\frac{\lambda_2}{\lambda_1} x^{(\lambda_2-\lambda_1)/\lambda_1} = 0.$$

4. Sketch the flow lines of a few of the systems

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} a & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \qquad a \neq 1$$

as $a \to 1$ (for instance, with $a = 2/4$, $3/4$, $5/4$ and $6/4$), and compare them with the flow lines of the limiting system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

What happens to the eigenvectors of the matrix $\begin{pmatrix} a & 1 \\ 0 & 1 \end{pmatrix}$ as $a \to 1$?

Solution. For any $a \neq 1$, the system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} a & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

is diagonalizable.

Let $A_a = \begin{pmatrix} a & 1 \\ 0 & 1 \end{pmatrix}$. The characteristic polynomial is

$$p_{A_a} = \det \begin{pmatrix} a - z & 1 \\ 0 & 1 - z \end{pmatrix} = (a - z)(1 - z),$$

and so the eigenvalues of A_a are $\lambda_1 = a$ and $\lambda_2 = 1$.

We find the corresponding eigenvectors —

For $\lambda_1 = a$,

$$\ker \begin{pmatrix} a - a & 1 \\ 0 & 1 - a \end{pmatrix} = \ker \begin{pmatrix} 0 & 1 \\ 0 & 1 - a \end{pmatrix},$$

so that

$$\mathbf{v_a} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

is an eigenvector for $\lambda_1 = a$.

For $\lambda_2 = 1$,

$$\ker \begin{pmatrix} a - 1 & 1 \\ 0 & 1 - 1 \end{pmatrix} = \ker \begin{pmatrix} a - 1 & 1 \\ 0 & 0 \end{pmatrix},$$

so that

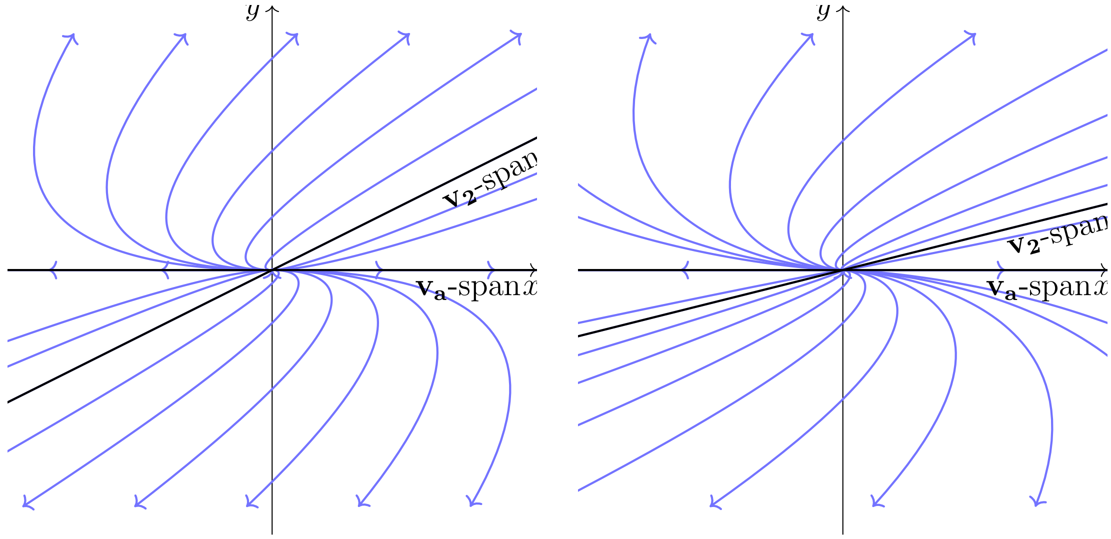
$$\mathbf{v_2} = \begin{pmatrix} 1 \\ -(a - 1) \end{pmatrix} = \begin{pmatrix} 1 \\ 1 - a \end{pmatrix}$$

is an eigenvector for $\lambda_2 = 1$.

Because a and 1 are both positive (for a sufficiently close to 1), the flow lines will form a stable node.

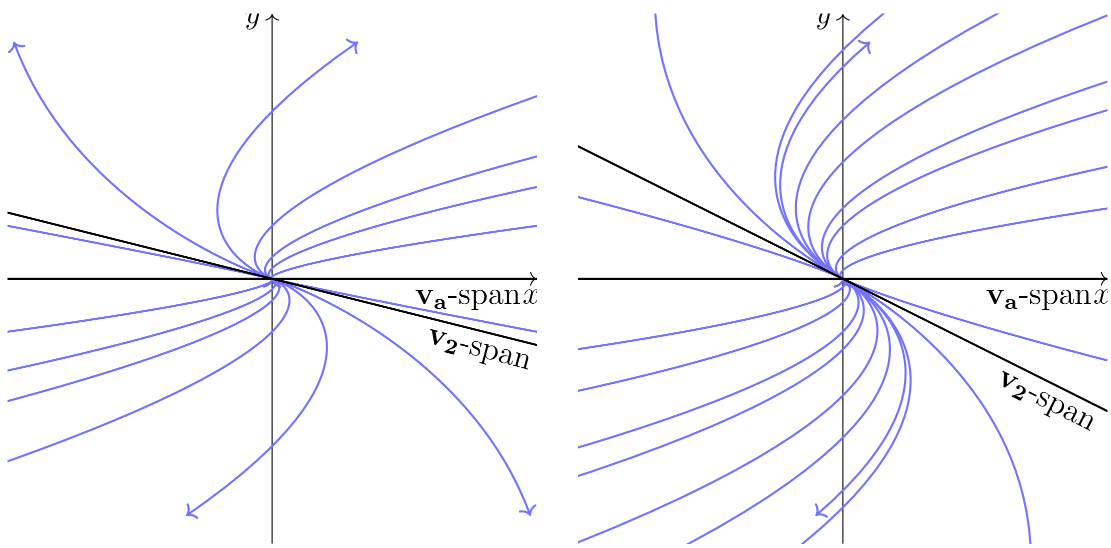
For $a < 1$, the flow lines will tend to being tangent with the $\mathbf{v_a}$-span as $t \to -\infty$, and tend to being parallel with $\mathbf{v_2}$ as $t \to \infty$.

For $a = 2/4$ (left) and $a = 3/4$ (right), the flow lines look as follows —

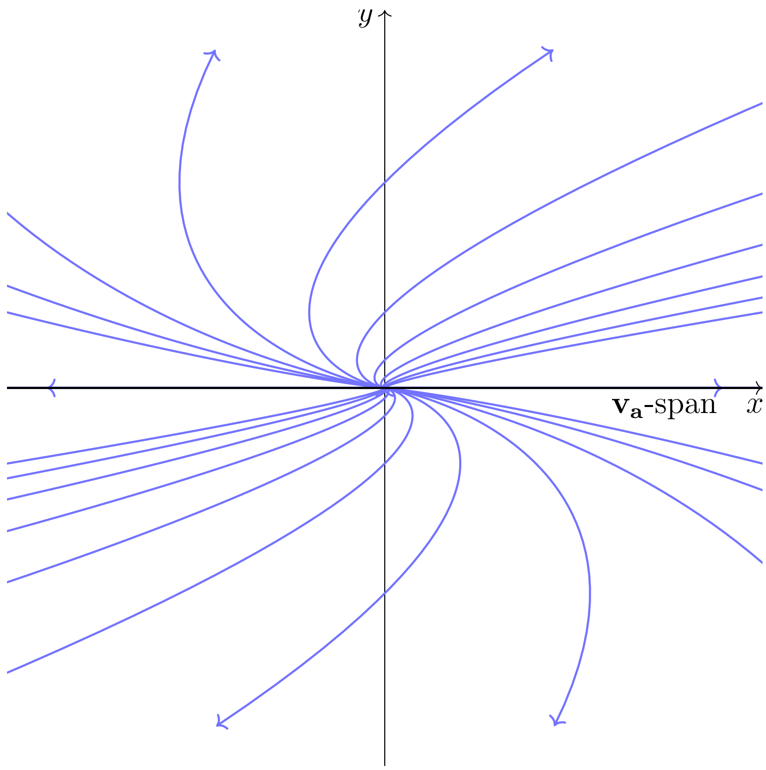


For $a > 1$, the flow lines will tend to being tangent with the $\mathbf{v_2}$-span as $t \to -\infty$, and tend to being parallel with $\mathbf{v_a}$ as $t \to \infty$.

For $a = 5/4$ (left) and $a = 6/4$ (right):



Finally, when $a = 1$, A_a has only one eigenvalue $\lambda = 1$ (of algebraic multiplicity 2). The two-dimensional subspace of $\mathbb{R}^2$ (so, all of $\mathbb{R}^2$) spanned by the eigenvectors in the $a \neq 1$ case collapses to a one-dimensional span (namely, the x-axis). The picture of the flow lines in this case is in some sense (that can be made precise) a limit of the previous pictures —


y
$\mathbf{v_a}$-span
x