Mthe 237 — Problem Set 09 Solutions

1. It is sometimes possible to find the curves traced out by solutions[1] of a first-order system of the form

$$\begin{aligned}\dot{x} &= v_1(x,y)\\ \dot{y} &= v_2(x,y)\end{aligned} \tag{1}$$

without finding the solutions of the system. The method is to make use of the identity

$$\frac{dy/dt}{dx/dt} = \frac{\dot{y}}{\dot{x}} = \frac{dy}{dx}, \quad \dot{x} \neq 0$$

that holds whenever the solution $t \mapsto (x(t), y(t))$ lies on the graph of a (differentiable) function $y(x)$ (as we have seen in the first lecture!). Thereby, we obtain the differential equation

$$\frac{dy}{dx} = \frac{v_2(x,y)}{v_1(x,y)} \tag{2}$$

that is satisfied by any function whose graph contains a solution of the system (1).

i) We have seen that the solutions of the system

$$\begin{aligned}\dot{x} &= x\\ \dot{y} &= y\end{aligned} \qquad \text{are} \qquad t \mapsto e^t \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}, \quad t \in \mathbb{R},$$

where $x(0) = x_0,\ y(0) = y_0$ are the initial conditions. In other words, the solutions are paths that trace out open rays starting at the origin and going radially outward to infinity, as well as the equilibrium solution $t \mapsto (0,0)$.

Solve equation (2) for this system, and check that the flow lines (with the exception of the two vertical ones) lie on graphs of the solutions of (2).

ii) Similarly, we have seen that the solutions of the system

$$\begin{aligned}\dot{x} &= -y\\ \dot{y} &= x\end{aligned} \qquad \text{are the circles} \qquad t \mapsto (A\cos(t+\phi),\ A\sin(t+\phi)), \quad t \in \mathbb{R},$$

where A and ϕ are determined by the initial conditions $x(0) = x_0,\ y(0) = y_0$ via $A = \sqrt{x_0^2 + y_0^2},\ \tan\phi = y_0/x_0$. Solve equation (2) for this system, and check that your findings are consistent our knowledge of the flow lines.

iii) Solve equation (2) for the following three systems

$$\begin{aligned}\dot{x} &= y\\ \dot{y} &= x\end{aligned}, \qquad \begin{aligned}\dot{x} &= 1 + 2y\\ \dot{y} &= 1 + 3x^2\end{aligned}, \qquad \begin{aligned}\dot{x} &= 2x^6y - 8x^4y^3 + 6x^2y^5\\ \dot{y} &= 8x^3y^4 - 6x^5y^2 - 2xy^6\end{aligned},$$

and sketch a few of the solutions (either graphs of solutions, or curves that are implicit solutions) — you can use a computer to help with the sketches (you are not asked to solve the three systems).

[1] More precisely, find curves containing the curves traced out by solutions.

Solution. i) The system of equations is

$$\dot{x} = x$$
$$\dot{y} = y,$$

and eq. (2) for this system is

$$\frac{dy}{dx} = \frac{y}{x}.$$

This is a separable equation. Separating variables,

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x}$$

integrating,

$$\ln|y| = \ln|x| + C.$$

Taking exponentials of both sides,

$$y(x) = Cx, \qquad C \in \mathbb{R}.$$

The graphs of solutions are a family of lines through the origin, which is consistent with the picture of flow lines as radial rays.

We can check that if $x_0 \neq 0$, then the flow line

$$t \mapsto e^t \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

lies on the graph of

$$y(x) = \frac{y_0}{x_0}x.$$

Indeed,

$$e^t y_0 = \frac{y_0}{x_0}e^t x_0.$$

ii) Eq. (2) for the system

$$\dot{x} = -y$$
$$\dot{y} = x$$

is

$$\frac{dy}{dx} = -\frac{x}{y}.$$

This is again separable.

$$y\frac{dy}{dx} = -x,$$

so that

$$\frac{y^2}{2} = -\frac{x^2}{2} + C,$$

and

$$x^2 + y^2 = C.$$

These are equations of circles, which is consistent with our picture of flow lines — we can check that for the flow line

$$t \mapsto \begin{pmatrix} A\cos(t+\phi) \\ A\sin(t+\phi) \end{pmatrix},$$

we have

$$(A\cos(t+\phi))^2 + (A\sin(t+\phi))^2 = A^2(\cos^2(t+\phi) + \sin^2(t+\phi)) = A^2.$$

iii) For the system

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= x, \end{aligned}$$

eq. (2) is

$$\frac{dy}{dx} = \frac{x}{y}.$$

Separable, as in the previous two cases —

$$y\frac{dy}{dx} = x,$$

so that

$$\frac{y^2}{2} = \frac{x^2}{2} + C,$$

or

$$x^2 - y^2 = C.$$

You may recognize these as equations of hyperbolas.

For the system

$$\begin{aligned} \dot{x} &= 1 + 2y \\ \dot{y} &= 1 + 3x^2, \end{aligned}$$

eq. (2) is

$$\frac{dy}{dx} = \frac{1+3x^2}{1+2y}.$$

Once again, this is separable. We get

$$(1+2y)\frac{dy}{dx} = 1 + 3x^2,$$

so that

$$y + y^2 = x + x^3 + C.$$

Finally, for the system

$$\dot{x} = 2x^6y - 8x^4y^3 + 6x^2y^5$$
$$\dot{y} = 8x^3y^4 - 6x^5y^2 - 2xy^6,$$

eq. (2) becomes

$$\frac{dy}{dx} = \frac{8x^3y^4 - 6x^5y^2 - 2xy^6}{2x^6y - 8x^4y^3 + 6x^2y^5}.$$

Multiplying by $2x^6y - 8x^4y^3 + 6x^2y^5$ and rearranging, we obtain

$$(6x^5y^2 - 8x^3y^4 + 2xy^6) + (2x^6y - 8x^4y^3 + 6x^2y^5)\frac{dy}{dx} = 0.$$

This is an exact equation (indeed,

$$\frac{\partial}{\partial y}(6x^5y^2 - 8x^3y^4 + 2xy^6) = 12x^5y - 32x^3y^3 + 12xy^5 = \frac{\partial}{\partial x}(2x^6y - 8x^4y^3 + 6x^2y^5)).$$

Integrating $M(x,y) = 6x^5y^2 - 8x^3y^4 + 2xy^6$ with respect to x, we get

$$x^6y^2 - 2x^4y^4 + x^2y^6 + h(y),$$

for some function $h(y)$ of y only. Taking the partial of this result with respect to y, we see that $h'(y) = 0$, hence implicit solutions are the level curves

$$x^6y^2 - 2x^4y^4 + x^2y^6 = (x^3y + xy^3)^2 = C.$$

The level curves describe a spider-web!

2. i) Write down a first-order system

$$\dot{\mathbf{x}} = A\mathbf{x} \tag{3}$$

that is equivalent to the equation

$$\frac{d^ry}{dt^r} + a_{r-1}\frac{d^{r-1}y}{dt^{r-1}} + \cdots + a_1\frac{dy}{dt} + a_0y = 0, \qquad a_j \in \mathbb{R}. \tag{4}$$

ii) Let $p_A(z) = \det(A - zI)$ be the characteristic polynomial of the linear map (or matrix) A in (3). Show that

$$p_A(z) = (-1)^r\chi(z),$$

where $\chi(z) = z^r + a_{r-1}z^{r-1} + \cdots + a_1z + a_0$ is the characteristic polynomial of (4). Conclude that the roots of $\chi(z)$ are exactly the eigenvalues of A, with the same multiplicities.

Suggestion: Try the cases $r = 2, 3, 4$ first to help see how to do the computation in general.

Solution. i) Rearranging, we have

$$\frac{d^ry}{dy^r} = -a_{r-1}\frac{d^{r-1}y}{dt^{r-1}} - \cdots - a_1\frac{dy}{dt} - a_0y,$$

and an equivalent first-order system (by a Proposition from class) is

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= x_3 \\
&\vdots \\
\dot{x}_{r-1} &= x_r \\
\dot{x}_r &= -a_{r-1}x_r - \cdots - a_1 x_2 - a_0 x_1
\end{aligned}$$

(unfortunately the indexing is off by one.)

This system of equations may be written as a matrix equation:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_r \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{r-1} \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_r \end{pmatrix}.$$

In the matrix A, there are 1's above each diagonal term.

More compactly, we can write this system down as

$$\dot{\mathbf{x}} = A\mathbf{x}.$$

ii) The characteristic polynomial of the matrix A is

$$\det(A - zI) = \det \begin{pmatrix} -z & 1 & 0 & \cdots & 0 & 0 \\ 0 & -z & 1 & \cdots & 0 & 0 \\ 0 & 0 & -z & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -z & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{r-2} & -a_{r-1} - z \end{pmatrix}.$$

One of the properties of the determinant is that adding a multiple of a row (or column) to another row (or column, respectively) does not change the determinant.

To cancel out the $-z$ in the first column, let's add z times the second column to the first. Unfortunately, the matrix we obtain, namely

$$\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ -z^2 & -z & 1 & \cdots & 0 & 0 \\ 0 & 0 & -z & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -z & 1 \\ -a_0 - a_1 z & -a_1 & -a_2 & \cdots & -a_{r-2} & -a_{r-1} - z \end{pmatrix}$$

now has a nonzero entry in the second row. Fortunately, adding z^2 times the third column to the first cancels out the $-z^2$. We obtain

$$\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & -z & 1 & \cdots & 0 & 0 \\ -z^3 & 0 & -z & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -z & 1 \\ -a_0 - a_1 z - a_2 z^2 & -a_1 & -a_2 & \cdots & -a_{r-2} & -a_{r-1} - z \end{pmatrix}$$

which now has a nonzero entry in the third row.

Continue in this manner, adding z^3 times the fourth column to the first, and so on, until adding z^{r-1} times the last (rth) column to the first.

We obtain a matrix with only the last entry of the first column not equal to zero, and that entry is $-a_0 - a_1 z - a_2 z^2 - \cdots - a_{r-1} z^{r-1} - z^r = -\chi(z)$. The matrix is

$$\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & -z & 1 & \cdots & 0 & 0 \\ 0 & 0 & -z & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -z & 1 \\ -\chi(z) & -a_1 & -a_2 & \cdots & -a_{r-2} & -a_{r-1} - z \end{pmatrix}$$

This matrix has the same determinant as $\det(A - zI)$.

Expanding along the first column, the sign in front of the $-\chi(z)$ term in the Laplace[2] expansion is + if r is odd and − if r is even, so that

$$\det(A - zI) = (-1)^{r-1}(-\chi(z)) \det \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ -z & 1 & 0 & \cdots & 0 & 0 \\ 0 & -z & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & -z & 1 \end{pmatrix}.$$

This last matrix has only 1's on the diagonal and only 0's above the diagonal (such matrices are called *lower triangular*), so we can continually expand along the first row to see that the determinant is equal to 1. (More generally, the determinant of a lower triangular matrix is equal to the product of the diagonal entries, by the same argument of continually expanding along the first row.)

This then gives the desired conclusion:

$$\det(A - zI) = (-1)^r \chi(z).$$

[2] He's at it again, that Laplace!

Alternative argument. The characteristic polynomial of the matrix A is

$$\det(A - zI) = \det\begin{pmatrix} -z & 1 & 0 & \cdots & 0 & 0 \\ 0 & -z & 1 & \cdots & 0 & 0 \\ 0 & 0 & -z & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -z & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{r-2} & -a_{r-1} - z \end{pmatrix}.$$

Expand this determinant along the bottom row.

The sign in front of the $(-a_0)$ term in the Laplace expansion is + if r is odd and − if r is even, so we can write the first term of the expansion as

$$(-1)^{r-1}(-a_0)\det\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ -z & 1 & 0 & \cdots & 0 \\ 0 & -z & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} + \text{terms 2 through } r \text{ of the expansion.}$$

The matrix we obtain in the expansion is lower-triangular once again, so its determinant is equal to 1.

So, the first term of the Laplace expansion is equal to $(-1)^r a_0$.

Now, the sign of the second term of the expansion will be the opposite from that of the first term. So we can continue the expansion as follows

$$(-1)^r a_0 + (-1)^r(-a_1)\det\begin{pmatrix} -z & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & -z & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} + \text{terms 3 through } r \text{ of the expansion.}$$

By the same principle that the determinant of a lower-triangular matrix is the product of its diagonal entries (or expanding continuously along the first row), we see that the determinant of the matrix is $-z$. Therefore, the second term of the expansion is

$$(-1)^r(-a_1)(-z) = (-1)^r a_1 z.$$

Doing one more term, we will have

$$(-1)^r(a_0 + a_1 z) + (-1)^{r-1}(-a_2)\det\begin{pmatrix} -z & 1 & 0 & 0 & \cdots & 0 \\ 0 & -z & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & -z & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{pmatrix} + \text{terms 4 through } r \text{ of the expansion.}$$

Expanding down along the columns with $(-z)$'s and then along rows with 1's, we see that the determinant of this matrix is $(-z)(-z)\cdot 1\cdots 1 = z^2$.

This pattern will continue: the $k+1$-st term of the Laplace expansion along the bottom row will have $(-z)$ in the diagonal entries of the first k columns (with 0's below the $(-z)$ entry), and 1's in the diagonal entries of the other columns (with 0's to the right of the 1 entry). Therefore, the contribution of the $k+1$-st term of the expansion will be $(-1)^r a_k z^k$, except the last term will contribute $(-1)^r(a_{r-1}+z)z^{r-1} = (-1)^r(a_{r-1}z^{r-1}+z^r)$.

Adding the terms of the expansion together, we obtain

$$(-1)^r(a_0 + a_1 z + a_2 z^2 + \cdots + a_{r-1}z^{r-1} + z^r) = (-1)^r \chi(z).$$

3. The Laplace transform method applies just as well to systems of differential equations. Solve the first-order system

$$\begin{aligned}\dot{x} &= x - 5y \\ \dot{y} &= x - y\end{aligned}, \qquad x(0)=1,\ y(0)=0, \tag{5}$$

as follows:

i) Take Laplace transforms of both expressions, obtaining a system of (algebraic) equations in $\mathscr{L}[x](s)$ and $\mathscr{L}[y](s)$.

ii) Solve the system of equations from part i) for $\mathscr{L}[x](s)$ and $\mathscr{L}[y](s)$.

iii) Take inverse Laplace transforms to obtain the solution $t \mapsto (x(t), y(t))$ of the system (5).

Solution. We follow the three steps above implicitly in this solution.

Taking Laplace transforms of both differential equations in the system (5), we obtain the system of equations

$$\begin{aligned}\mathscr{L}[\dot{x}](s) &= \mathscr{L}[x](s) - 5\mathscr{L}[y](s) \\ \mathscr{L}[\dot{y}](s) &= \mathscr{L}[x](s) - \mathscr{L}[y](s).\end{aligned}$$

The transforms of the derivatives of x and y become, using the initial conditions,

$$\begin{aligned}s\mathscr{L}[x](s) - 1 &= \mathscr{L}[x](s) - 5\mathscr{L}[y](s) \\ s\mathscr{L}[y](s) - 0 &= \mathscr{L}[x](s) - \mathscr{L}[y](s).\end{aligned}$$

Let us introduce the (standard) notation $\mathscr{L}[x](s) = X(s)$ and $\mathscr{L}[y](s) = Y(s)$ (the capital letters denote the transforms and the small letters denote the corresponding functions in the t-domain). The system of equations written in this notation is

$$\begin{aligned}sX(s) - 1 &= X(s) - 5Y(s) \\ sY(s) &= X(s) - Y(s).\end{aligned}$$

We can solve the second equation for $Y(s)$ in terms of $X(s)$:

$$(s+1)Y(s) = X(s) \qquad \text{so} \qquad Y(s) = \frac{X(s)}{s+1}.$$

Then, plugging this into the first equation, we obtain

$$sX(s) - 1 = X(s) - 5\frac{X(s)}{s+1}.$$

Rearranging,

$$\left((s-1) + \frac{5}{s+1}\right)X(s) = 1,$$

so that

$$\frac{s^2 - 1 + 5}{s+1}X(s) = 1,$$

and so

$$X(s) = \frac{s+1}{s^2+4} = \frac{s}{s^2+4} + \frac{1}{2}\frac{2}{s^2+4}.$$

Remembering the transforms $\cos(\omega t) = s/(s^2+\omega^2)$ and $\sin(\omega t) = \omega/(s^2+\omega^2)$, we see that

$$x(t) = \mathscr{L}^{-1}\left[\frac{s}{s^2+4}\right] + \mathscr{L}^{-1}\left[\frac{1}{2}\frac{2}{s^2+4}\right] = \cos(2t) + \frac{1}{2}\sin(2t).$$

Then,

$$Y(s) = \frac{X(s)}{s+1} = \frac{1}{s^2+4} = \frac{1}{2}\frac{2}{s^2+4},$$

so that

$$y(t) = \frac{1}{2}\sin(2t).$$

The solution trajectory satisfying the initial conditions is

$$t \mapsto (x(t), y(t)) = (\cos(2t) + \frac{1}{2}\sin(2t),\ \frac{1}{2}\sin(2t)).$$

4. Let $A = \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}$. Compute the matrix exponential $\exp(A)$.

Solution. We begin by computing the eigenvalues of A and their corresponding eigenspaces. The characteristic polynomial of A factors as

$$\det(A - zI) = \det\begin{pmatrix} 2-z & 3 \\ 1 & 4-z \end{pmatrix} = (2-z)(4-z) - 3 = z^2 - 6z + 5 = (z-1)(z-5).$$

Therefore, the eigenvalues of A are $\lambda_1 = 1$ and $\lambda_2 = 5$. Since these are distinct, the matrix will be diagonalizable.

To find an eigenvector corresponding to $\lambda_1 = 1$, we try to solve

$$A\mathbf{v} = 1 \cdot \mathbf{v} = \mathbf{v} \quad (\mathbf{v} \neq 0),$$

or, equivalently, find a nonzero vector such that

$$(A - I)\mathbf{v} = 0.$$

By definition, such a vector will lie in

$$\ker(A - I).$$

We have

$$A - I = \begin{pmatrix} 1 & 3 \\ 1 & 3 \end{pmatrix}.$$

The two rows impose the conditions

$$x + 3y = 0$$

on the coordinates of a vector $\mathbf{v} = (x, y)$ to be in the kernel of $A - I$ (the second row is redundant, being equal to the first row).

By inspection, we see that the vector

$$\mathbf{v_1} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$$

lies in the kernel. We take $\mathbf{v_1}$ as the eigenvector spanning the eigenspace of λ_1 (any nonzero multiple of $\mathbf{v_1}$ would also work.)

Similarly, to find an eigenvector corresponding to $\lambda_2 = 5$, we look in the kernel

$$\ker A - 5I.$$

We have

$$A - 5I = \begin{pmatrix} -3 & 3 \\ 1 & -1 \end{pmatrix},$$

so the condition for a vector to lie in the eigenspace corresponding to λ_2 is

$$x - y = 0.$$

We see that we can take

$$\mathbf{v_2} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

as a vector spanning the eigenspace of λ_2.

In the ordered basis $\{\lambda_1, \lambda_2\}$, the matrix A becomes

$$\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix}.$$

Let

$$Q = \begin{pmatrix} \mathbf{v_1} & \mathbf{v_2} \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ -1 & 1 \end{pmatrix}$$

be the change of basis matrix from the eigenbasis to the original basis of A.

By the formula for the inverse of a 2×2 matrix (or otherwise),

$$Q^{-1} = \frac{1}{\det Q} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \frac{1}{4} \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}.$$

Then,

$$A = Q\begin{pmatrix}1 & 0\\ 0 & 5\end{pmatrix}Q^{-1},$$

and so

$$\exp(A) = Q\begin{pmatrix}e & 0\\ 0 & e^5\end{pmatrix}Q^{-1} = \begin{pmatrix}3 & 1\\ -1 & 1\end{pmatrix}\begin{pmatrix}e & 0\\ 0 & e^5\end{pmatrix}\frac{1}{4}\begin{pmatrix}1 & -1\\ 1 & 3\end{pmatrix} = \frac{1}{4}\begin{pmatrix}3 & 1\\ -1 & 1\end{pmatrix}\begin{pmatrix}e & -e\\ e^5 & 3e^5\end{pmatrix} = \frac{1}{4}\begin{pmatrix}3e + e^5 & -3e + 3e^5\\ -e + e^5 & e + 3e^5\end{pmatrix}.$$