Mthe 237 — Problem Set 08 Solutions

1. Solve the following equations using the Laplace transform method.

i) $\dfrac{dy}{dt} - y = e^t, \quad y(0) = 1.$

ii) $\dfrac{d^2y}{dt^2} + 4\dfrac{dy}{dt} + 4y = 0, \qquad y(0) = 1, \quad \dfrac{dy}{dt}(0) = 1.$

iii) $\dfrac{d^2y}{dt^2} - 2\dfrac{dy}{dt} + 5y = 0, \qquad y(0) = 2, \quad \dfrac{dy}{dt}(0) = 4.$

iv) $\dfrac{d^3y}{dt^3} - 2\dfrac{d^2y}{dt^2} + \dfrac{dy}{dt} = 2e^t + 2t, \qquad y(0) = 0, \quad \dfrac{dy}{dt}(0) = 0, \quad \dfrac{d^2y}{dt^2}(0) = 0.$

Optional Problem. Also solve the two equations from Problem Set 6 using the Laplace transform:

v) $\dfrac{d^2y}{dt^2} + \dfrac{dy}{dt} - 2y = 2t, \qquad y(0) = 0, \quad \dfrac{dy}{dt}(0) = 1.$

vi) $\dfrac{d^2y}{dt^2} + 16y = t^2 + \sin(4t), \qquad y(0) = \dfrac{127}{128}, \quad \dfrac{dy}{dt}(0) = \dfrac{7}{8}.$

These will require more algebraic manipulations to get the expression for $\mathscr{L}[y](s)$ into a form where the inverse transform can be found by inspection.

Solution. i) Taking Laplace transforms of both sides, we get

$$\mathscr{L}[y'](s) - \mathscr{L}[y](s) = \mathscr{L}[e^t](s)$$

or

$$(s\mathscr{L}[y](s) - y(0)) - \mathscr{L}[y](s) = \frac{1}{s-1}.$$

Using the initial condition $y(0) = 1$, and grouping like terms, we get

$$(s-1)\mathscr{L}[y](s) - 1 = \frac{1}{s-1}$$

so that, solving for $\mathscr{L}[y](s)$,

$$\mathscr{L}[y](s) = \frac{1}{s-1} + \frac{1}{(s-1)^2}.$$

By inspection, the inverse Laplace transform is

$$y(t) = e^t + te^t.$$

ii) Taking Laplace transforms of both sides, we get

$$\mathscr{L}[y''](s) + 4\mathscr{L}[y'](s) + 4\mathscr{L}[y](s) = 0$$

or

$$\left(s^2\mathscr{L}[y](s) - sy(0) - y'(0)\right) + 4\left(s\mathscr{L}[y](s) - y(0)\right) + 4\mathscr{L}[y](s) = 0.$$

Grouping like terms and applying the initial conditions, we get

$$(s^2 + 4s + 4)\mathscr{L}[y](s) - s - 5 = 0,$$

so that

$$\mathscr{L}[y](s) = \frac{s+5}{s^2+4s+4} = \frac{s+5}{(s+2)^2} = \frac{1}{s+2} + \frac{3}{(s+2)^2}.$$

Taking the inverse Laplace transform by inspection, we find

$$y(t) = e^{-2t} + 3t\,e^{-2t}.$$

iii) Taking Laplace transforms of both sides, we get

$$\mathscr{L}[y''](s) - 2\mathscr{L}[y'](s) + 5\mathscr{L}[y](s) = 0,$$

or

$$\left(s^2\mathscr{L}[y](s) - sy(0) - y'(0)\right) - 2\left(s\mathscr{L}[y](s) - y(0)\right) + 5\mathscr{L}[y](s) = 0.$$

Grouping like terms and applying the initial conditions, we have

$$(s^2 - 2s + 5)\mathscr{L}[y](s) - 2s = 0,$$

so that

$$\mathscr{L}[y](s) = \frac{2s}{s^2-2s+5} = \frac{2s}{(s-1)^2+4} = \frac{2(s-1)+2}{(s-1)^2+4} = 2\frac{s-1}{(s-1)^2+4} + \frac{2}{(s-1)^2+4}.$$

Taking the inverse Laplace transform,

$$y(t) = 2e^t\cos(2t) + e^t\sin(2t).$$

Remark. It is also acceptable to do this problem by breaking up $\frac{1}{s^2-2s+5}$ into *complex* partial fractions, although strictly speaking we have not defined the Laplace transform for complex-valued functions. If one proceeds this way, however, after obtaining complex-valued exponentials in the t-domain, it is necessary to apply Euler's formula to the results to obtain real-valued solutions.

For the sake of illustration, here is how this approach would work for this problem. We have

$$\begin{aligned}\frac{2s}{s^2-2s+5} &= \frac{2s}{(s-(1+2i))(s-(1-2i))} \\ &= \frac{A}{s-(1+2i)} + \frac{B}{s-(1-2i)} \\ &= \frac{A(s-(1-2i)) + B(s-(1+2i))}{s^2-2s+5}\end{aligned}$$

Comparing coefficients of s and 1, we get the system of equations

$$\begin{aligned}A + B &= 2 \\ -A(1-2i) - B(1+2i) &= 0\end{aligned}$$

Let's solve it using matrix methods from linear algebra. We have

$$\left(\begin{array}{cc|c} 1 & 1 & 2 \\ -1+2i & -1-2i & 0 \end{array}\right) \xrightarrow{R2\cdot(-1-2i)} \left(\begin{array}{cc|c} 1 & 1 & 2 \\ 5 & -3+4i & 0 \end{array}\right)$$
$$\xrightarrow{R2-5R1} \left(\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & -8+4i & -10 \end{array}\right)$$
$$\xrightarrow{R2\cdot 1/(-8+4i)} \left(\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & 1+\frac{i}{2} \end{array}\right) \quad \left(\frac{-10}{-8+4i} = \frac{-5}{-4+2i} = \frac{-5(-4-2i)}{(-4+2i)(-4-2i)} = \frac{20+10i}{16+4} = 1+\frac{i}{2}.\right)$$
$$\xrightarrow{R1-R2} \left(\begin{array}{cc|c} 1 & 0 & 1-\frac{i}{2} \\ 0 & 1 & 1+\frac{i}{2} \end{array}\right).$$

Therefore, $A = 1 - \frac{i}{2}$ and $B = 1 + \frac{i}{2}$. We have

$$\mathscr{L}[y](s) = \frac{2s}{s^2 - 2s + 5} = \left(1 - \frac{i}{2}\right)\frac{1}{1-(1+2i)} + \left(1 + \frac{i}{2}\right)\frac{1}{1-(1-2i)}.$$

Taking on faith that the inverse Laplace transform of $\frac{1}{s-a}$ is e^{at} even when a is complex, we then have

$$\begin{aligned} y(t) &= \left(1 - \frac{i}{2}\right)e^{(1+2i)t} + \left(1 + \frac{i}{2}\right)e^{(1-2i)t} \\ &= \left(e^{(1+2i)t} + e^{(1-2i)t}\right) - \frac{i}{2}\left(e^{(1+2i)t} - e^{(1-2i)t}\right). \end{aligned}$$

By Euler's formula,

$$\frac{e^{\sigma+i\omega} + e^{\sigma-i\omega}}{2} = e^{\sigma}\cos(\omega) \qquad \text{and} \qquad \frac{e^{\sigma+i\omega} - e^{\sigma-i\omega}}{2i} = e^{\sigma}\sin(\omega).$$

Therefore, using the fact that $i = -\frac{1}{i}$,

$$y(t) = 2e^t\cos(2t) + e^t\sin(2t).$$

iv) Taking Laplace transforms of both sides, we get

$$\mathscr{L}[y'''](s) - 2\mathscr{L}[y''](s) + \mathscr{L}[y'](s) = \frac{2}{s-1} + \frac{2}{s^2}.$$

Since the initial conditions are all equal to zero, the left side is particularly simple:

$$s^3\mathscr{L}[y](s) - 2s^2\mathscr{L}[y](s) + s\mathscr{L}[y](s) = s(s^2-2s+1)\mathscr{L}[y](s) = s(s-1)^2\mathscr{L}[y](s) = \frac{2}{s-1} + \frac{2}{s^2}.$$

Therefore,

$$\mathscr{L}[y](s) = \frac{2}{s(s-1)^3} + \frac{2}{s^3(s-1)^2} = \frac{2s^2 + 2(s-1)}{s^3(s-1)^3} = \frac{2s^2+2s-2}{s^3(s-1)^3}.$$

We find the partial fraction decomposition of the last term —

$$\begin{aligned} \frac{2s^2+2s-2}{s^3(s-1)^3} &= \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s^3} + \frac{D}{s-1} + \frac{E}{(s-1)^2} + \frac{F}{(s-1)^3} \\ &= \frac{As^2(s-1)^3 + Bs(s-1)^3 + C(s-1)^3 + Ds^3(s-1)^2 + Ds^3(s-1) + Es^3}{s^3(s-1)^3} \\ &= \frac{A(s^5-3s^4+3s^3-s^2) + B(s^4-3s^3+3s^2-s) + C(s^3-3s^2+3s-1) + D(s^5-2s^4+s^3) + E(s^4-s^3) + Fs^3}{s^3(s-1)^3} \end{aligned}$$

We get the system of linear equations

$$\begin{aligned} A + D &= 0 \\ -3A + B - 2D + E &= 0 \\ 3A - 3B + C + D - E + F &= 0 \\ -A + 3B - 3C &= 2 \\ -B + 3C &= 2 \\ -C &= -2 \end{aligned}$$

From the last equation, $C = 2$. From the fifth equation, then, $B = 4$. From the fourth equation, $A = 4$. From the first equation, $A = -D$, so that $D = -4$. Then, from the second equation, $E = 0$. Finally, from the third equation, $F = 2$.

Alternatively, bringing the coefficient matrix to rref,

$$\left(\begin{array}{cccccc|c} 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ -3 & 1 & 0 & -2 & 1 & 0 & 0 \\ 3 & -3 & 1 & 1 & -1 & 1 & 0 \\ -1 & 3 & -3 & 0 & 0 & 0 & 2 \\ 0 & -1 & 3 & 0 & 0 & 0 & 2 \\ 0 & 0 & -1 & 0 & 0 & 0 & -2 \end{array}\right) \xrightarrow{R2+3R1,\ R3-3R1, R4+R1} \left(\begin{array}{cccccc|c} 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & -3 & 1 & -2 & -1 & 1 & 0 \\ 0 & 3 & -3 & 1 & 0 & 0 & 2 \\ 0 & -1 & 3 & 0 & 0 & 0 & 2 \\ 0 & 0 & -1 & 0 & 0 & 0 & -2 \end{array}\right)$$

$$\xrightarrow{R3+3R2,\ R4-3R2,\ R5+R2} \left(\begin{array}{cccccc|c} 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 2 & 1 & 0 \\ 0 & 0 & -3 & -2 & -3 & 0 & 2 \\ 0 & 0 & 3 & 1 & 1 & 0 & 2 \\ 0 & 0 & -1 & 0 & 0 & 0 & -2 \end{array}\right)$$

$$\xrightarrow{R4+3R3,\ R6-3R3,\ R6+R3} \left(\begin{array}{cccccc|c} 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 & 3 & 3 & 2 \\ 0 & 0 & 0 & -2 & -5 & -3 & 2 \\ 0 & 0 & 0 & 1 & 2 & 1 & -2 \end{array}\right)$$

$$\xrightarrow{R5+2R4,\ R6-R4} \left(\begin{array}{cccccc|c} 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 & 3 & 3 & 2 \\ 0 & 0 & 0 & 0 & 1 & 3 & 6 \\ 0 & 0 & 0 & 0 & -1 & -2 & -4 \end{array}\right)$$

$$\xrightarrow{R6+R5} \left(\begin{array}{cccccc|c} 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 & 3 & 3 & 2 \\ 0 & 0 & 0 & 0 & 1 & 3 & 6 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 \end{array}\right)$$

$$\xrightarrow{R3-R6,\ R4-3R6, R5-3R6} \left(\begin{array}{cccccc|c} 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 2 & 0 & -2 \\ 0 & 0 & 0 & 1 & 3 & 0 & -4 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 \end{array}\right)$$

$$\xrightarrow{R2-R5,\ R3-2R5,\ R4-3R5} \left(\begin{array}{cccccc|c} 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & -2 \\ 0 & 0 & 0 & 1 & 0 & 0 & -4 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 \end{array}\right)$$

$$\xrightarrow{R1-R4,\ R2-R4,\ R3-R4} \left(\begin{array}{cccccc|c} 1 & 0 & 0 & 0 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 & 0 & -4 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 \end{array}\right).$$

Therefore,

$$\frac{2s^2+2s-2}{s^3(s-1)^3} = \frac{4}{s} + \frac{4}{s^2} + \frac{2}{s^3} - \frac{4}{s-1} + \frac{2}{(s-1)^3}.$$

Taking the inverse Laplace transform,

$$y(t) = 4 + 4t + t^2 - 4e^t + t^2 e^t.$$

v) Taking Laplace transforms of both sides,

$$\mathscr{L}[y''](s) + \mathscr{L}[y'](s) - 2\mathscr{L}[y](s) = \frac{2}{s^2},$$

or

$$(s^2\mathscr{L}[y](s) - sy(0) - y'(0)) + (s\mathscr{L}[y](s) - y(0)) - 2\mathscr{L}[y](s) = \frac{2}{s^2}.$$

Applying the initial conditions $y(0) = 0,\ y'(0) = 1$,

$$(s^2 + s - 2)\mathscr{L}[y](s) - 1 = \frac{2}{s^2},$$

so that

$$\mathscr{L}[y](s) = \frac{1}{s^2+s-2} + \frac{2}{s^2(s^2+s-2)} = \frac{1}{(s-1)(s+2)} + \frac{2}{s^2(s-1)(s+2)} = \frac{s^2+2}{s^2(s-1)(s+2)}.$$

We find the partial fraction decomposition of the last term —

$$\begin{aligned} \frac{s^2+2}{s^2(s-1)(s+2)} &= \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-1} + \frac{D}{s+2} \\ &= \frac{As(s^2+s-2) + B(s^2+s-2) + Cs^2(s+2) + Ds^2(s-1)}{s^2(s-1)(s+2)} \\ &= \frac{A(s^3+s^2-2s) + B(s^2+s-2) + C(s^3+2s^2) + D(s^3-s^2)}{s^2(s-1)(s+2)} \end{aligned}$$

We get the system of linear equations

$$\begin{aligned} A + C + D &= 0 \\ A + B + 2C - D &= 1 \\ -2A + B &= 0 \\ -2B &= 2 \end{aligned}$$

From the last equation, $B = -1$. Then, from the third, $A = -1/2$. Adding the first two, $2A+B+3C = 1$, so $C = 1$. Finally, from the first, $D = -1/2$.

$$\frac{s^2+2}{s^2(s-1)(s+2)} = -\frac{1}{2}\frac{1}{s} - \frac{1}{s^2} + \frac{1}{s-1} - \frac{1}{2}\frac{1}{s+2}.$$

Taking the inverse Laplace transform,

$$y(t) = -\frac{1}{2} - t + e^t - \frac{1}{2}e^{-2t}.$$

vi) Taking Laplace transforms of both sides,

$$\mathscr{L}[y''](s) + 16\mathscr{L}[y](s) = \frac{2}{s^3} + \frac{4}{s^2+16},$$

or

$$(s^2+16)\mathscr{L}[y](s) - \frac{127s}{128} - \frac{7}{8} = \frac{2}{s^3} + \frac{4}{s^2+16}.$$

So

$$\mathscr{L}[y](s) = \frac{7}{8}\frac{1}{s^2+16} + \frac{127}{128}\frac{s}{s^2+16} + \frac{2}{s^3(s^2+16)} + \frac{4}{(s^2+16)^2}.$$

We find the partial fraction decomposition of the third term —

$$\begin{aligned} \frac{2}{s^3(s^2+16)} &= \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s^3} + \frac{Ds+E}{s^2+16} \\ &= \frac{As^2(s^2+16) + Bs(s^2+16) + C(s^2+16) + Ds^4 + Es^3}{s^3(s^2+16)} \\ &= \frac{A(s^4+16s^2) + B(s^3+16s) + C(s^2+16) + Ds^4 + Es^3}{s^3(s^2+16)}. \end{aligned}$$

The solution is $A = -1/128,\ B = 0,\ C = 1/8,\ D = 1/128,\ E = 0$. We get (after rearranging slightly)

$$\mathscr{L}[y](s) = -\frac{1}{128}\frac{1}{s} + \frac{1}{8}\frac{1}{s^3} + \frac{s}{s^2+16} + \frac{7}{8}\frac{1}{s^2+16} + \frac{4}{(s^2+16)^2}.$$

Because of the term $\frac{1}{(s^2+16)^2}$, we recognize some transforms of functions of type $t\cos$ and $t\sin$. We have (from Problem 2)

$$\mathscr{L}[t\cos(\omega t)](s) = \frac{s^2-\omega^2}{(s^2+\omega^2)^2} \qquad \text{and} \qquad \mathscr{L}[t\sin(\omega t)](s) = \frac{2\omega s}{(s^2+\omega^2)^2}.$$

We would like to bring this expression to the form where we can recognize the transforms by inspection. One trick that works is to write

$$\frac{7}{8}\frac{1}{s^2+16} = \frac{1}{s^2+16} - \frac{1}{8}\frac{1}{s^2+16} = \frac{1}{s^2+16} - \frac{1}{8}\frac{s^2+16}{(s^2+16)^2}.$$

Then

$$-\frac{1}{8}\frac{s^2+16}{(s^2+16)^2} + \frac{4}{(s^2+16)^2} = -\frac{1}{8}\frac{s^2-16}{(s^2+16)^2} \quad (!).$$

So that

$$sL[y](s) = -\frac{1}{128}\frac{1}{s} + \frac{1}{16}\frac{2}{s^3} + \frac{s}{s^2+16} + \frac{1}{s^2+16} - \frac{1}{8}\frac{s^2-16}{(s^2+16)^2}$$

and

$$y(t) = -\frac{1}{128} + \frac{1}{16}t^2 + \cos(4t) + \frac{1}{4}\sin(4t) - \frac{1}{8}t\cos(4t),$$

in agreement with the solution found in Problem Set 06.

2. i) Show that $\int_0^\infty f(t)e^{-st}\,dt$ converges for $s = s_0 - \sigma$ if and only if $\int_0^\infty \left(e^{\sigma t}f(t)\right)e^{-st}\,dt$ converges for $s = s_0$.

Hence, show that if the domain of $\mathscr{L}[f]$ is all $s > a$, then the domain of $\mathscr{L}[e^{\sigma t}f]$ is all $s > a + \sigma$, and we have

$$\mathscr{L}[e^{\sigma t}f](s) = \mathscr{L}[f](s-\sigma) \quad \text{for all} \quad s > a + \sigma.$$

ii) Taking the four transforms

$$\mathscr{L}[\cos(\omega t)](s) = \frac{s}{s^2+\omega^2},\ s > 0, \qquad \mathscr{L}[\sin(\omega t)](s) = \frac{\omega}{s^2+\omega^2},\ s > 0,$$

$$\mathscr{L}[t\cos(\omega t)](s) = \frac{s^2-\omega^2}{(s^2+\omega^2)^2},\ s > 0 \qquad \text{and} \qquad \mathscr{L}[t\sin(\omega t)](s) = \frac{2\omega s}{(s^2+\omega^2)^2},\ s > 0$$

as known, find

$$\mathscr{L}[e^{\sigma t}\cos(\omega t)], \quad \mathscr{L}[te^{\sigma t}\cos(\omega t)], \quad \mathscr{L}[e^{\sigma t}\sin(\omega t)] \quad \text{and} \quad \mathscr{L}[te^{\sigma t}\cos(\omega t)].$$

Solution. i) We have

$$\int_0^\infty (e^{\sigma t}f(t))e^{-st}\,dt = \int_0^\infty f(t)e^{-(s-\sigma)t}\,dt.$$

Therefore, $\int_0^\infty f(t)e^{-(s_0-\sigma)t}\,dt$ converges if and only if $\int_0^\infty (e^{\sigma t}f(t))e^{-s_0 t}\,dt$ converges. They are the same integral!

Now, observe that this implies that $\int_0^\infty f(t)\,e^{-st}\,dt$ converges for $s = s_0$ if and only if $\int_0^\infty (e^{\sigma t}f(t))\,e^{-st}\,dt$ converges for $s = s_0 + \sigma$.

Suppose that the domain of $\mathscr{L}[f]$ is all $s > a$, for some a (the other possibilities for the domain — $s \geq a$ and the empty set, can be handled similarly). This means that $\int_0^\infty f(t)\,e^{-st}\,dt$ converges if and only if $s > a$, by definition of the domain of $\mathscr{L}[f]$. By the observation of the previous paragraph, it follows that $\int_0^\infty (e^{\sigma t}f(t))\,e^{-st}\,dt$ converges if and only if $s > a + \sigma$. Therefore, the domain of $\mathscr{L}[e^{\sigma t}f]$ is all $s > a + \sigma$.

We have, using the definition of Laplace transform,

$$\begin{aligned}\mathscr{L}[e^{\sigma t}f](s) &= \int_0^\infty (e^{\sigma t}f(t))\,e^{-st}\,dt \\ &= \int_0^\infty f(t)e^{-(s-\sigma)t}\,dt \\ &= \mathscr{L}[f](s-\sigma) \qquad \text{for all } s > a + \sigma\end{aligned}$$

ii) Applying the first part, we have

$$\mathscr{L}[e^{\sigma t}\cos(\omega t)](s) = \mathscr{L}[\cos(\omega t)](s-\sigma) = \frac{(s-\sigma)}{(s-\sigma)^2+\omega^2}, \quad s>\sigma;$$
$$\mathscr{L}[t\,e^{\sigma t}\cos(\omega t)](s) = \mathscr{L}[t\,\cos(\omega t)](s-\sigma) = \frac{(s-\sigma)^2-\omega^2}{((s-\sigma)^2+\omega^2)^2}, \quad s>\sigma;$$
$$\mathscr{L}[e^{\sigma t}\sin(\omega t)](s) = \mathscr{L}[\sin(\omega t)](s-\sigma) = \frac{\omega}{(s-\sigma)^2+\omega^2}, \quad s>\sigma;$$
$$\mathscr{L}[t\,e^{\sigma t}\sin(\omega t)](s) = \mathscr{L}[t\,\sin(\omega t)](s-\sigma) = \frac{2\omega(s-\sigma)}{((s-\sigma)^2+\omega^2)^2}, \quad s>\sigma.$$

3. As a reminder, earlier in the term we defined the functions

$$\cosh(\omega t) = \frac{e^{\omega t}+e^{-\omega t}}{2} \qquad \text{and} \qquad \sinh(\omega t) = \frac{e^{\omega t}-e^{-\omega t}}{2}, \qquad t\in\mathbb{R}.$$

Find expressions for $\mathscr{L}[\cosh(\omega t)]$ and $\mathscr{L}[\sinh(\omega t)]$ in the following two ways:

i) Take as known the transform

$$\mathscr{L}[e^{\sigma t}](s) = \frac{1}{s-\sigma}, \quad s>0,$$

and use linearity of $\mathscr{L}$.

ii) Check that

$$\cosh(0)=1, \qquad \frac{d}{dt}\cosh(\omega t)\Big|_{t=0} = 0,$$

and

$$\sinh(0)=0, \qquad \frac{d}{dt}\sinh(\omega t)\Big|_{t=0} = \omega.$$

Then, find a linear homogeneous differential equation with constant coefficients that has cosh and sinh as solutions, apply $\mathscr{L}$ to both sides of the differential equation, and use the initial conditions found above.

(Although you do not have to show this, the domain of both $\mathscr{L}[\cosh(\omega t)]$ and $\mathscr{L}[\sinh(\omega t)]$ is equal to $s>|\omega|$, as may be tempting to guess from the domains of $e^{\omega t}$ and $e^{-\omega t}$.)

Optional Problem. Compute $\mathscr{L}[\cosh(\omega t)]$ and $\mathscr{L}[\sinh(\omega t)]$ from the definition, and verify the claim just made about their domains.

Solution. i) By linearity of the Laplace transform, we have

$$\begin{aligned}\mathscr{L}[\cosh(\omega t)](s) &= \frac{1}{2}\mathscr{L}[e^{\omega t}](s) + \frac{1}{2}\mathscr{L}[e^{-\omega t}](s)\\ &= \frac{1}{2}\frac{1}{s-\omega} + \frac{1}{2}\frac{1}{s+\omega}\\ &= \frac{1}{2}\frac{(s+\omega)+(s-\omega)}{s^2-\omega^2}\\ &= \frac{s}{s^2-\omega^2},\end{aligned}$$

for all $s > \max(\omega, -\omega)$ (since the domains of $\mathscr{L}[e^{\omega t}]$ and $\mathscr{L}[e^{-\omega t}]$ are $s > \omega$ and $s > -\omega$, respectively), and

$$\begin{aligned}\mathscr{L}[\sinh(\omega t)](s) &= \frac{1}{2}\mathscr{L}[e^{\omega t}](s) - \frac{1}{2}\mathscr{L}[e^{-\omega t}](s)\\ &= \frac{1}{2}\frac{1}{s-\omega} - \frac{1}{2}\frac{1}{s+\omega}\\ &= \frac{1}{2}\frac{(s+\omega)-(s-\omega)}{s^2-\omega^2}\\ &= \frac{\omega}{s^2-\omega^2},\end{aligned}$$

for all $s > \max(\omega, -\omega)$.

Remarks. The domains of $\mathscr{L}[\cosh(\omega t)]$ and $\mathscr{L}[\sinh(\omega t)]$ could be larger than the regions where we can apply the linearity properties of $\mathscr{L}$, but in this example it turns out that they are not.

It is worth noticing how similar these expressions are to the Laplace transforms of cos and sin.

ii) Since $\cosh(\omega t)$ and $\sinh(\omega t)$ are linear combinations of $e^{\omega t}$ and $e^{-\omega t}$, they are in fact quasipolynomials! The problem of finding an equation for cosh and sinh is equivalent to finding an annihilator of cosh and sinh, and we have previously (in Problem Set 06) developed some methods on how to find annihilators of linear combinations of functions.

The operator $\left(\frac{d}{dt} - \omega\right)$ is an annihilator of $e^{\omega t}$, and $\left(\frac{d}{dt} + \omega\right)$ is an annihilator of $e^{-\omega t}$. Therefore, the operator product

$$\left(\frac{d}{dt} - \omega\right)\left(\frac{d}{dt} + \omega\right) = \frac{d^2}{dt^2} - \omega^2$$

is an annihilator of any linear combination of $e^{\omega t}$ and $e^{-\omega t}$. In particular, it is an annihilator of both $\cosh(\omega t)$ and $\sinh(\omega t)$. Therefore, $\cosh(\omega t)$ and $\sinh(\omega t)$ are solutions of the differential equation

$$\frac{d^2y}{dt^2} - \omega^2 y = 0. \tag{1}$$

Applying the Laplace transform to both sides of this equation, we get

$$\mathscr{L}[\frac{d^2y}{dt^2}](s) - \omega^2\mathscr{L}[y](s) = 0.$$

Therefore,

$$s^2\mathscr{L}[y](s) - sy(0) - y'(0) - \omega^2\mathscr{L}[y](s) = 0,$$

so that

$$(s^2-\omega^2)\mathscr{L}[y](s) = sy(0) + y'(0).$$

Finally,

$$\mathscr{L}[y](s) = \frac{sy(0)+y'(0)}{s^2-\omega^2}, \tag{2}$$

for any solution y of (1).

Computing the derivatives of $\cosh(\omega t)$ and $\sinh(\omega t)$, we have

$$(\cosh(\omega t))' = \left(\frac{e^{\omega t} + e^{-\omega t}}{2}\right)' = \frac{\omega e^{\omega t} - \omega e^{-\omega t}}{2} = \omega\left(\frac{e^{\omega t} - e^{-\omega t}}{2}\right) = \omega \sinh(\omega t),$$

and

$$(\sinh(\omega t))' = \left(\frac{e^{\omega t} - e^{-\omega t}}{2}\right)' = \frac{\omega e^{\omega t} + \omega e^{-\omega t}}{2} = \omega \cosh(\omega t).$$

Moreover, $\cosh(0) = \frac{e^0+e^0}{2} = 1$ and $\sinh(0) = \frac{e^0-e^0}{2} = 0$, so that

$$\cosh(0) = 1, \qquad \frac{d}{dt}\cosh(\omega t)\bigg|_{t=0} = \omega \sinh(0) = 0,$$

and

$$\sinh(0) = 0, \qquad \frac{d}{dt}\sinh(\omega t)\bigg|_{t=0} = \omega \cosh(0) = \omega.$$

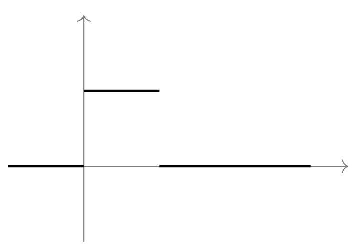
Finally, using result (2) with these initial conditions, we see that the Laplace transforms of $\cosh(\omega t)$ and $\sinh(\omega t)$ are

$$\mathscr{L}[\cosh(\omega t)](s) = \frac{s \cdot 1 + 0}{s^2 - \omega^2} = \frac{s}{s^2 - \omega^2}$$

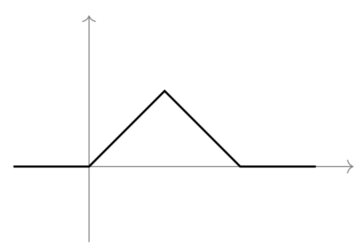
and

$$\mathscr{L}[\sinh(\omega t)](s) = \frac{s \cdot 0 + \omega}{s^2 - \omega^2} = \frac{\omega}{s^2 - \omega^2}.$$

4. Let f and g denote the following two functions.



and



$$f(t) = \begin{cases} 1 & 0 \le t \le 1 \\ 0 & \text{otherwise} \end{cases} \qquad\qquad g(t) = \begin{cases} t & 0 \le t \le 1 \\ 2 - t & 1 \le t \le 2 \\ 0 & \text{otherwise} \end{cases}$$

i) Show that the convolution $f * f$ of f with itself is equal to g, in the following two ways.

First, compute $f * f$ using the definition of convolution

$$(f * f)(t) = \int_{-\infty}^{\infty} f(u) f(t-u)\, du$$

and check that the result is equal to g.

Second, write both f and g as an expression involving step functions, and apply the property $\mathscr{L}[f * f] = \mathscr{L}[f]\mathscr{L}[f]$ of the Laplace transform.

ii) Compute the convolution $(2f) * g$ of $2f$ with g in the same two ways as part i).

Optional Problem. Investigate the motion of a simple harmonic oscillator starting from rest

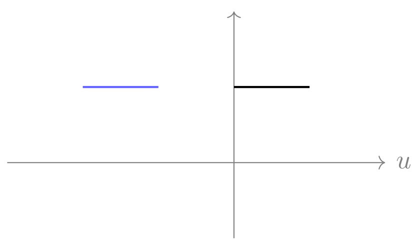
$$\frac{d^2y}{dt^2} + y = F(t), \qquad y(0) = 0, \quad \frac{dy}{dt}(0) = 0,$$

subject to forcing functions $f(t)$, $g(t)$ and $((2f) * g)(t)$.

Solution. i) The computation breaks up into a number of cases based on the intersection of the graph of $f(u)$ with the graph of $f(t-u)$. We can think of the latter as the graph of f, but reflected about the y-axis, and translated by t to the right (or, more generally, translated by t along the positive u-direction).

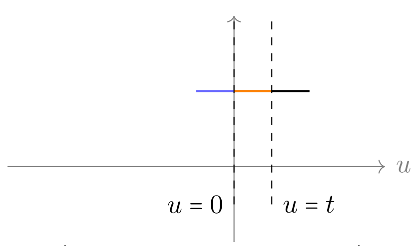
We adopt the convention that the fixed graph of $f(u)$ is in black, and the sliding graph of $f(t-u)$ is in blue. When the two graphs overlap along a line segment, we shall denote the overlap in orange. Moreover, we omit the graphs of both where they are equal to zero.

Case $t < 0$: The regions where $f(u)$ and $f(t-u)$ are nonzero do not overlap. Therefore, their product $f(u)f(t-u)$ is equal to 0 for all u, and



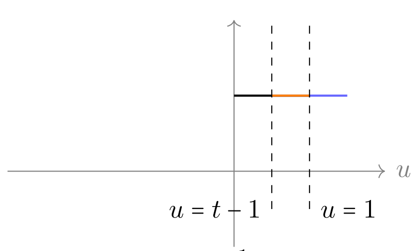
$$(f * f)(t) = \int_{-\infty}^{\infty} 0\,du = 0.$$

Case $0 \le t < 1$: The tip of $f(t-u)$ crosses over into the region where $f(u)$ is nonzero.



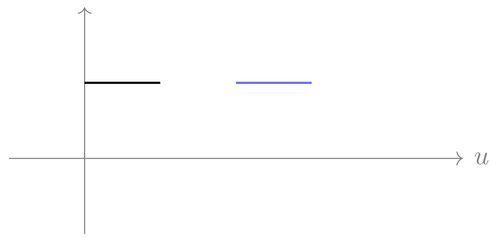
$$(f * f)(t) = \int_0^t f(u)f(t-u)\,du = \int_0^t 1\,du = u\big|_{u=0}^{u=t} = t.$$

Case $1 \le t < 2$: The tip of $f(t-u)$ passes the region where $f(u)$ is nonzero, but the tail of $f(t-u)$ still overlaps the region.



$$(f * f)(t) = \int_{t-1}^1 f(u)f(t-u)\,du = \int_{t-1}^1 1\,du = u\big|_{u=t-1}^{u=1} = 1 - (t-1) = 2 - t.$$

Case $t \geq 2$*:* The graph of $f(t-u)$ completely passes the region where $f(u)$ is nonzero, so the product $f(u)f(t-u)$ is again zero.



$$(f * f)(t) = 0.$$

In summary,

$$(f * f)(t) = \begin{cases} t & 0 \leq t \leq 1 \\ 2-t & 1 \leq t \leq 2 \\ 0 & \text{otherwise} \end{cases},$$

so that $(f * f)(t) = g(t)$ for all $t \in \mathbb{R}$.

Writing f using unit step functions, we have

$$f(t) = u_0(t) - u_1(t) \quad \text{for all } t.$$

Therefore,

$$\mathscr{L}[f](s) = \frac{e^{-0s}}{s} - \frac{e^{-1s}}{s} = \frac{1}{s} - \frac{e^{-s}}{s}.$$

By the convolution theorem,

$$\mathscr{L}[f * f](s) = \mathscr{L}[f](s)\,\mathscr{L}[f](s) = \left(\frac{1}{s} - \frac{e^{-s}}{s}\right)^2 = \frac{1}{s^2} - \frac{2}{s^2}e^{-s} + \frac{1}{s^2}e^{-2s}.$$

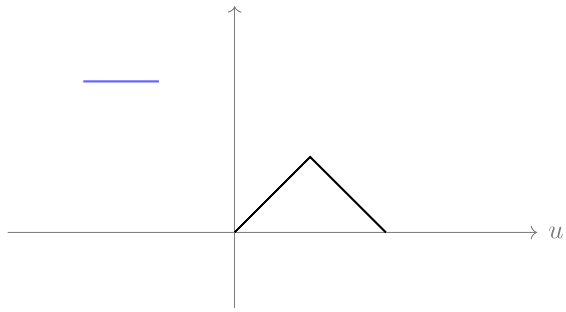
We can find the inverse Laplace transform using the property that if $Y(s) = \mathscr{L}[y](s)$, then $\mathscr{L}^{-1}[e^{-as}Y] = u_a(t)y(t-a)$. Remembering that $\mathscr{L}^{-1}[\frac{1}{s^2}] = t$, we have

$$\begin{aligned} (f * f)(t) &= u_0(t)t - 2u_1(t)(t-1) + u_2(t)(t-2) \\ &= \begin{cases} 0 & t < 0 \\ t & 0 \leq t < 1 \\ t - 2(t-1) = 2-t & 1 < t \leq 2 \\ (2-t) + (t-2) = 0 & t > 2 \end{cases} \end{aligned}$$

which once again recovers $g(t)$ for all $t \in \mathbb{R}$.

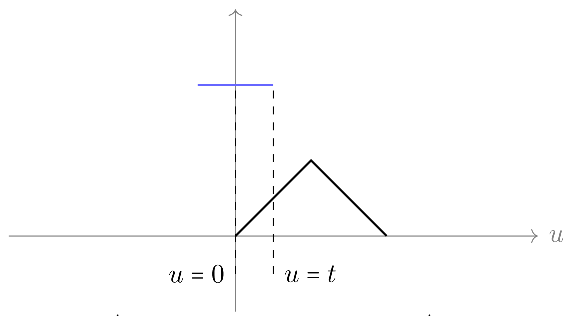
ii) By a property proved in class, $(2f * g) = (g * 2f)$. We compute the second instead, because it is somewhat simpler to imagine sliding a horizontal line segment. We proceed as in part i), with the same conventions.

Case $t < 0$*:* The regions where $g(u)$ and $2f(t-u)$ are nonzero do not overlap. Therefore, their product $g(u)\,2f(t-u)$ is equal to 0 for all u, and



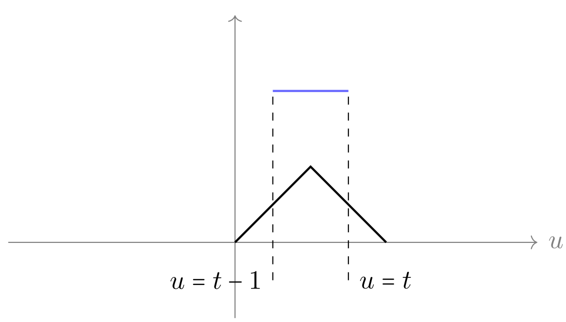
$$(g * 2f)(t) = \int_{-\infty}^{\infty} 0\,du = 0.$$

Case $0 \le t < 1$*:* The tip of the segment $2f(t-u)$ overlaps the region where $g(u)$ is nonzero.



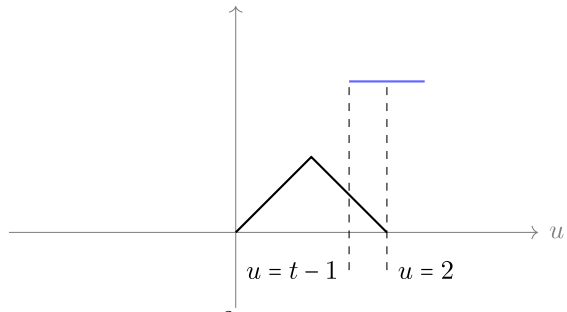
$$(g * 2f)(t) = \int_0^t g(u)\,2f(t-u)\,du = \int_0^t 2u\,du = u^2\Big|_{u=0}^{u=t} = t^2.$$

Case $1 \le t < 2$*:* The entire segment $2f(t-u)$ overlaps the region where $g(u)$ is nonzero.



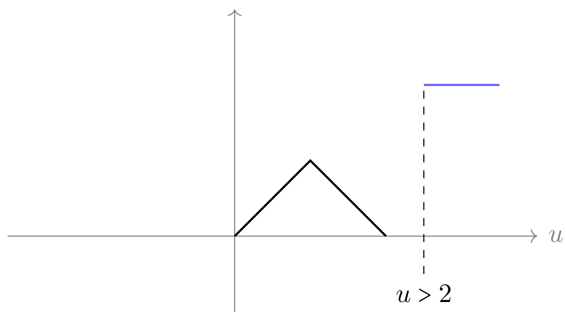
$$\begin{aligned}(g * 2f)(t) = \int_{t-1}^t g(u)\,2f(t-u)\,du &= \int_{t-1}^1 2u\,du + \int_1^t 2(2-u)\,du \\ &= u^2\Big|_{u=t-1}^{u=1} + (4u-u^2)\Big|_{u=1}^{u=t} \\ &= 1-(t-1)^2 + (4t-t^2) - (4-1) \\ &= -2t^2 + 6t - 3 \\ &= -2\left(t - \frac{3}{2}\right)^2 + \frac{15}{2}.\end{aligned}$$

Case $2 \le t < 3$*:* The tail of the segment $2f(t-u)$ overlaps the region where $g(u)$ is nonzero.



$$(g*2f)(t) = \int_{t-1}^{2} g(u)\,2f(t-u)\,du = \int_{t-1}^{2} 2(2-u)\,du = \left.(4u-u^2)\right|_{u=t-1}^{u=2} = t^2-6t+9 = (t-3)^2.$$

Case $t \geq 3$: The tail of the segment $2f(t-u)$ passes the region where $g(u)$ is nonzero.

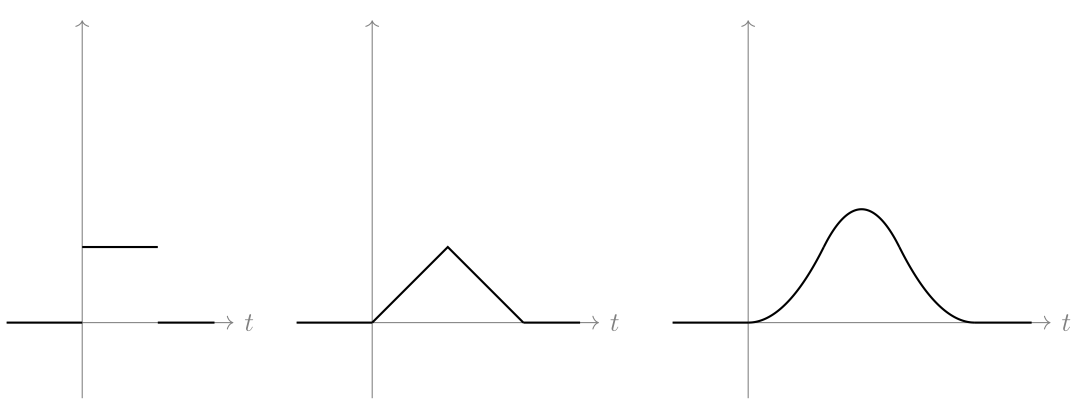


$$(g*2f)(t) = 0.$$

In summary,

$$(g*2f)(t) = \begin{cases} t^2 & 0 \leq t < 1 \\ -2t^2+6t-3 & 1 \leq t < 2 \\ t^2-6t+9 & 2 \leq t < 3 \\ 0 & \text{otherwise} \end{cases} = \begin{cases} t^2 & 0 \leq t < 1 \\ -2\left(t-\frac{3}{2}\right)^2+\frac{15}{2} & 1 \leq t < 2 \\ (t-3)^2 & 2 \leq t < 3 \\ 0 & \text{otherwise} \end{cases}.$$

The graph of $(g*2f)$ is a nice union of three parabolas. The function $(g*2f)$, which we can think of as $f*f*f$ (up to multiplying by 2), is now differentiable (whereas f was discontinuous, and $f*f$ was merely continuous).



f $\qquad$ $f*f$ $\qquad$ $f*f*f$

Finally, we know from the previous part that

$$\mathscr{L}[g](s) = \frac{1}{s^2}\left(1 - 2e^{-s} + e^{-2s}\right).$$

By the convolution theorem,

$$\begin{aligned}\mathscr{L}[g * 2f](s) = \mathscr{L}[g](s)\mathscr{L}[2f](s) &= \frac{1}{s^2}\left(1 - 2e^{-s} + e^{-2s}\right) 2\left(\frac{1}{s} - \frac{e^{-s}}{s}\right) \\ &= \frac{2}{s^3}\left(1 - 3e^{-s} + 3e^{-2s} - e^{-3s}\right).\end{aligned}$$

By the translation theorem (recalling that $\mathscr{L}[t^2](s) = 2/s^3$), we conclude that

$$(g * 2f)(t) = u_0(t)t^2 - 3u_1(t)(t-1)^2 + 3u_2(t)(t-2)^2 - u_3(t)(t-3)^2.$$

Writing out the pieces,

$$(g * 2f)(t) = \begin{cases} 0 & t < 0 \\ t^2 & 0 \le t < 1 \\ t^2 - 3(t-1)^2 = -2t^2 + 6t - 3 & 1 \le t < 2 \\ (-2t^2 + 6t - 3) + 3(t-2)^2 = t^2 - 6t + 9 & 2 \le t < 3 \\ (t^2 - 6t + 9) - (t-3)^2 = 0 & t \ge 3 \end{cases}.$$

This is the same answer as before!