Mthe 237 — Problem Set 07 Solutions

1. Solve the following equations using the method of variation of parameters.

i) $\dfrac{dy}{dt} - \dfrac{2t}{t^2+1}y = 1, \quad y(0) = 0.$ (The integral $\int \frac{ds}{s^2+1} = \arctan(s) + C$ may be useful.)

ii) $\dfrac{d^2y}{dt^2} + 2\dfrac{dy}{dt} + y = \dfrac{e^{-t}}{t}, \quad t > 0 \qquad y(1) = 0, \ \dfrac{dy}{dt}(1) = -e^{-1}.$

iii) $2t^2\dfrac{d^2y}{dt^2} + 3t\dfrac{dy}{dt} - y = \dfrac{1}{t}, \quad t > 0, \qquad y(1) = 0, \ \dfrac{dy}{dt}(1) = \dfrac{11}{6},$
given that $\phi_1 = t^{1/2}$ and $\phi_2 = t^{-1}$ are solutions of the associated homogeneous equation.

Solution. i) To use variation of parameters to find a solution, we first need to find a (nonzero) solution of the associated homogeneous equation

$$\frac{dy}{dt} - \frac{2t}{t^2+1}y = 0.$$

As are all linear homogeneous equations of first order, the equation is separable. Separating variables, we obtain

$$\frac{1}{y}\frac{dy}{dt} = \frac{2t}{t^2+1}.$$

Integrating both sides, we get

$$\ln(|y|) = \ln(t^2+1) + C.$$

For the purposes of separation of variables, it is enough to find a single nonzero solution, so we may as well take $C = 0$, and choose the positive branch of the absolute value:

$$\phi(t) = t^2 + 1.$$

Next, by the separation of variables method, we need to find a function u whose derivative u' satisfies the linear equation

$$\phi u' = F = 1.$$

Dividing both sides by ϕ, we obtain

$$\frac{du}{dt} = \frac{1}{t^2+1},$$

so that, integrating with respect to t,

$$u(t) = \int \frac{dt}{t^2+1} = \arctan(t).$$

(We may omit the arbitrary constant, since we are looking for any particular solution at the moment.) A particular solution produced by separation of variables is then

$$\phi_p(t) = \phi u = (t^2+1)\arctan(t).$$

The affine space of all solutions is

$$(t^2+1)\arctan(t) + b\,(t^2+1), \qquad b \in \mathbb{R}.$$

Imposing the initial condition $y(0) = 0$, we see

$$(0^2+1)\arctan(0) + b\,(1) = 0,$$

so that

$$b = 0$$

and the solution satisfying the initial condition $y(0) = 0$ is

$$(t^2+1)\arctan(t), \qquad t \in \mathbb{R}.$$

ii) We begin by finding two linearly independent solutions of the associated homogeneous equation

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = 0.$$

This has characteristic polynomial $\chi(z) = z^2 + 2z + 1 = (z+1)^2$. The characteristic polynomial has a double root at $z = -1$, so we can take

$$\phi_1(t) = e^{-t}, \quad \phi_2(t) = te^{-t}$$

as the two linearly independent solutions.

The Wronskian of ϕ_1, ϕ_2 is

$$W(\phi_1, \phi_2)(t) = \det\begin{pmatrix} e^{-t} & te^{-t} \\ -e^{-t} & (1-t)e^{-t} \end{pmatrix} = (1-t)e^{-2t} - (-te^{-2t}) = e^{-2t}.$$

Following variation of parameters, we now look for functions u_1, u_2 whose derivatives satisfy the linear system of equations

$$\begin{pmatrix} \phi_1 & \phi_2 \\ \phi_1' & \phi_2' \end{pmatrix}\begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 0 \\ e^{-t}/t \end{pmatrix}.$$

This system can be solved in a number of different ways. Here we use Cramer's rule. We have

$$u_1' = \frac{\det\begin{pmatrix} 0 & te^{-t} \\ e^{-t}/t & (1-t)e^{-t} \end{pmatrix}}{W(\phi_1, \phi_2)(t)} = \frac{-e^{-2t}}{e^{-2t}} = -1,$$

and

$$u_2' = \frac{\det\begin{pmatrix} e^{-t} & 0 \\ -e^{-t} & e^{-t}/t \end{pmatrix}}{W(\phi_1, \phi_2)(t)} = \frac{e^{-2t}/t}{e^{-2t}} = \frac{1}{t},$$

so that

$$u_1(t) = \int -1\,dt = -t$$

and

$$u_2(t) = \int \frac{dt}{t} = \ln(t)$$

(we have omitted the absolute value on t because the domain of the equation was restricted to $t > 0$ in the problem statement).

The affine space of all solutions is then

$$-te^{-t} + t\ln(t)e^{-t} + b_1 e^{-t} + b_2 te^{-t}, \qquad b_1, b_2 \in \mathbb{R}.$$

We can absorb the first term $-te^{-t}$ of the particular solution into the homogeneous piece $b_2\, t\, e^{-t}$, so that the affine space of solutions is

$$t\ln(t)e^{-t} + b_1 e^{-t} + b_2 te^{-t}, \qquad b_1, b_2 \in \mathbb{R}.$$

The derivative of such a solution is

$$(\ln(t)e^{-t} + e^{-t} - t\ln(t)e^{-t}) - b_1 e^{-t} + b_2 e^{-t} - b_2 te^{-t}.$$

Imposing the initial conditions $y(1) = 0$, $\frac{dy}{dt} = -e^{-1}$, we obtain

$$\begin{aligned} 0 + (b_1 + b_2)e^{-1} &= 0, \\ (0 + e^{-1} + 0) + (-b_1 + b_2 - b_2)e^{-1} &= -e^{-1}, \end{aligned}$$

or

$$\begin{aligned} b_1 + b_2 &= 0, \\ -b_1 &= -2. \end{aligned}$$

This has solutions $b_1 = 2$, $b_2 = -2$. Therefore, the solution to the differential equation satisfying the initial conditions is

$$t\ln(t)e^{-t} + 2e^{-t} - 2te^{-t}, \quad t > 0.$$

iii) We begin by bringing the equation to standard form (with coefficient 1 in front of the highest derivative) —

$$\frac{d^2y}{dt^2} + \frac{3}{2t}\frac{dy}{dt} - \frac{1}{2t^2}y = \frac{1}{2t^3}, \quad t > 0.$$

We are provided with the solutions $\phi_1(t) = t^{1/2}$ and $\phi_2(t) = t^{-1}$ of the associated homogeneous equation.

We can check that these in fact are solutions of the associated homogeneous equation —

$$\phi_1' = \frac{1}{2}t^{-1/2}, \qquad \phi_1'' = -\frac{1}{4}t^{-3/2},$$

so

$$\phi_1'' + \frac{3}{2t}\phi_1' - \frac{1}{2t^2}\phi_1 = -\frac{1}{4}t^{-3/2} + \frac{3}{2t}\frac{1}{2}t^{-1/2} - \frac{1}{2t^2}t^{1/2} = \left(-\frac{1}{4} + \frac{3}{4} - \frac{1}{2}\right)t^{-3/2} = 0,$$

and

$$\phi_2' = -t^{-2}, \qquad \phi_2'' = 2t^{-3},$$

so

$$\phi_2'' + \frac{3}{2t}\phi_2' - \frac{1}{2t^2}\phi_2 = 2t^{-3} + \frac{3}{2t}(-t^{-2}) - \frac{1}{2t^2}t^{-1} = \left(2 - \frac{3}{2} - \frac{1}{2}\right)t^{-3} = 0.$$

The Wronskian of ϕ_1 and ϕ_2 is

$$W(\phi_1, \phi_2)(t) = \det\begin{pmatrix} t^{1/2} & t^{-1} \\ \frac{1}{2}t^{-1/2} & -t^{-2} \end{pmatrix} = -t^{-3/2} - \frac{1}{2}t^{-3/2} = -\frac{3}{2}t^{-3/2}.$$

Following variation of parameters, we look for functions u_1, u_2 whose derivatives satisfy the linear system

$$\begin{pmatrix} \phi_1 & \phi_2 \\ \phi_1' & \phi_2' \end{pmatrix}\begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 0 \\ 1/(2t^3) \end{pmatrix}.$$

We use Cramer's rule.

$$u_1' = \frac{\det\begin{pmatrix} 0 & t^{-1} \\ \frac{1}{2t^3} & -t^{-2} \end{pmatrix}}{W(\phi_1, \phi_2)(t)} = \frac{-\frac{1}{2}t^{-4}}{-\frac{3}{2}t^{-3/2}} = \frac{1}{3}t^{-5/2}$$

and

$$u_2' = \frac{\det\begin{pmatrix} t^{1/2} & 0 \\ \frac{1}{2}t^{-1/2} & \frac{1}{2t^3} \end{pmatrix}}{-\frac{3}{2}t^{-3/2}} = \frac{\frac{1}{2}t^{-5/2}}{-\frac{3}{2}t^{-3/2}} = -\frac{1}{3}t^{-1}.$$

Therefore,

$$u_1(t) = \int \frac{1}{3}t^{-5/2}\,dt = -\frac{2}{9}t^{-3/2}$$

and

$$u_2(t) = \int -\frac{1}{3}\frac{dt}{t} = -\frac{1}{3}\ln(t).$$

The affine space of all solutions is

$$-\frac{2}{9}t^{-1} - \frac{1}{3}t^{-1}\ln(t) + b_1\,t^{1/2} + b_2\,t^{-1}, \qquad b_1, b_2 \in \mathbb{R}.$$

We can absorb the term $-2/9t^{-1}$ of the particular solution into b_2, so that the affine space of all solutions is

$$-\frac{1}{3}\,t^{-1}\ln(t) + b_1t^{1/2} + b_2t^{-1}, \qquad b_1, b_2 \in \mathbb{R}.$$

A derivative of such a solution is

$$\frac{1}{3}t^{-2}\ln(t) - \frac{1}{3}t^{-2} + \frac{b_1}{2}t^{-1/2} - b_2t^{-2}.$$

Imposing the initial conditions $y(1) = 0,\ \frac{dy}{dt}(1) = \frac{11}{6}$, we get

$$0 + b_1 + b_2 = 0,$$
$$0 - \frac{1}{3} + \frac{b_1}{2} - b_2 = \frac{11}{6},$$

which is solved to get $b_1 = 13/9,\ b_2 = -13/9$.

The solution of the differential equation satisfying the initial conditions is

$$-\frac{1}{3}t^{-1}\ln(t) + \frac{13}{9}t^{1/2} - \frac{13}{9}t^{-1}, \qquad t > 0.$$

2. A first-order differential equation of the form

$$\frac{dy}{dt} + a(t)y = F(t)y^k, \quad k \neq 0, 1 \text{ real}, \quad y \neq 0 \tag{1}$$

is called a *Bernoulli equation.*

Although Bernoulli equations are nonlinear (unless $k = 0$ or $k = 1$, which is the reason for excluding these exponents), they can always be converted to linear equations by a change of variable.

i) Let $y(t)$ be a solution of the Bernoulli equation (1). Let $v(t) = y(t)^{1-k}$. Show that $\frac{1}{1-k}\frac{dv}{dt} = y^{-k}\frac{dy}{dt}$. Then, show that $v(t)$ is a solution of the differential equation

$$\frac{dv}{dt} + (1-k)a(t)v = (1-k)F(t), \tag{2}$$

which is now linear.

(Conversely, one can similarly show that if $v(t)$ is a solution of (2), then $y(t) = v(t)^{1/(1-k)}$ is a solution of (1).)

ii) Solve the equation

$$\frac{dy}{dt} = \epsilon y - \sigma y^3, \quad \epsilon > 0,\ \sigma > 0, \quad y(0) = \sqrt{2\epsilon/\sigma}$$

by recognizing it as a Bernoulli equation and making the above change of variable.

iii) Solve the equation

$$\frac{dy}{dt} + y = ty^3, \quad y(0) = 1.$$

Solution. i) Differentiating $v(t) = y(t)^{1-k}$ with respect to t, we get

$$\frac{dv}{dt} = (1-k)y^{-k}\frac{dy}{dt}$$

by the chain rule. Dividing both sides by $(1-k)$ shows the first claim.

Now, dividing the Bernoulli equation (1) by y^k, we get

$$y^{-k}\frac{dy}{dt} + a(t)y^{1-k} = F(t),$$

which, rewritten in terms of v, is

$$\frac{1}{1-k}\frac{dv}{dt} + a(t)v = F(t).$$

Finally, multiplying both sides by $(1-k)$ to put the equation into standard form, we get

$$\frac{dv}{dt} + (1-k)a(t)v = (1-k)F(t).$$

ii) The equation of y is

$$\frac{dy}{dt} - \epsilon y = -\sigma y^3,$$

so that by part i), the function $v(t) = y(t)^{1-3} = \frac{1}{y^2}$ is a solution of

$$\frac{dv}{dt} + 2\epsilon v = 2\sigma.$$

This is linear, and in fact has constant coefficients. The associated homogeneous equation is

$$\frac{dv}{dt} + 2\epsilon v = 0.$$

The characteristic polynomial is $\chi(z) = z + 2\epsilon$, which has a single real root at $z = -2\epsilon$, so a nonzero solution of the associated homogeneous equation is $\phi(t) = e^{-2\epsilon t}$.

Now, following variation of parameters, we look for a function u whose derivative satisfies

$$u' = \frac{F}{\phi} = \frac{2\sigma}{e^{-2\epsilon t}} = 2\sigma e^{2\epsilon t}.$$

Integrating, we find

$$u(t) = \int 2\sigma e^{2\epsilon t}\,dt = \frac{\sigma}{\epsilon}e^{2\epsilon t}.$$

Therefore, the affine space of all solutions is

$$v(t) = \frac{\sigma}{\epsilon}e^{2\epsilon t}e^{-2\epsilon t} + b\,e^{-2\epsilon t} = \frac{\sigma}{\epsilon} + b\,e^{-2\epsilon t}, \qquad b \in \mathbb{R}.$$

The induced initial condition on v is

$$v(0) = \frac{1}{y(0)^2} = \frac{1}{2\epsilon/\sigma} = \frac{\sigma}{2\epsilon}.$$

We find b:

$$\frac{\sigma}{\epsilon} + b = \frac{\sigma}{2\epsilon},$$

so $b = -\frac{\sigma}{2\epsilon}$. The solution satisfying the initial condition is

$$v(t) = \frac{\sigma}{\epsilon}\left(1 - \frac{e^{-2\epsilon t}}{2}\right).$$

Finally, converting the solution back to y, $y(t) = v(t)^{-1/2}$ (we take the positive square root to match the initial condition), so

$$y(t) = \left[\frac{\sigma}{\epsilon}\left(1 - \frac{e^{-2\epsilon t}}{2}\right)\right]^{-1/2}.$$

We can determine the domain of the solution: because we require the solution to be real-valued (and can't divide by 0), the argument of the square root must be positive. This happens if and only if

$$1 - \frac{e^{-2\epsilon t}}{2} > 0 \qquad \text{or} \qquad t > -\frac{\ln(2)}{2\epsilon}.$$

iii) The equation of y is

$$\frac{dy}{dt} + y = ty^3,$$

so that $v(t) = 1/y(t)^2$ is a solution of

$$\frac{dv}{dt} - 2v = -2t.$$

The corresponding homogeneous equation for v is

$$\frac{dv}{dt} - 2v = 0,$$

which has characteristic polynomial $\chi(z) = z - 2$, with a real root at $z = 2$. Therefore, we can take $\phi = e^{2t}$ as a nonzero solution of the associated homogeneous equation. Following variation of parameters, we now look for u whose derivative satisfies

$$u' = \frac{-2t}{e^{2t}} = -2te^{-2t}.$$

Integrating by parts, we have

$$u(t) = \int -2te^{-2t}\,dt = te^{-2t} - \int e^{-2t}\,dt = te^{-2t} + \frac{1}{2}e^{-2t},$$

so that the affine space of all solutions is

$$v(t) = u(t)\phi(t) + b\,\phi(t) = t + \frac{1}{2} + b\,e^{2t} \qquad b \in \mathbb{R}.$$

The induced initial condition on v is $v(0) = 1/y(0)^2 = 1$, so

$$0 + \frac{1}{2} + b = 1, \qquad \text{and} \qquad b = \frac{1}{2}.$$

The solution in terms of v is

$$v(t) = t + \frac{1}{2}(1 + e^{2t}).$$

Converting back to y, we have

$$y(t) = \frac{1}{\sqrt{t + \frac{1}{2}(1 + e^{2t})}}$$

The domain of the solution is all t that satisfy

$$t + \frac{1}{2}(1 + e^{2t}) > 0 \qquad \text{or} \qquad 2t + 1 + e^{2t} > 0.$$

This expression goes to $-\infty$ as $t \to -\infty$ and ∞ as $t \to \infty$, so it has at least one zero by the intermediate value theorem. Moreover, its derivative is $2 + 2e^{2t}$, which is positive for all t, so that the zero of $2t + 1 + e^{2t}$ is unique. Let ξ be this zero. Then the domain is $t > \xi$. Since $-2 + 1 + e^{-2t} < 0$ and $0 + 1 + e^0 > 0$, we know that $-1 < \xi < 0$, for instance. Using Newton's method, for example, we can estimate that $\xi \approx -0.639$.

3. In this problem, we look at resonance in a simple harmonic oscillator.

Consider an undamped spring with spring constant k hanging vertically, with one end fixed and the other end attached to a mass m. As derived in Problem Set 04, the equation of motion of the mass about its rest point is

$$\frac{d^2y}{dt^2} + \omega_0^2 y = 0,$$

where ω_0 is the frequency of the two solutions, $\cos(\omega_0 t)$ and $\sin(\omega_0 t)$, called the *natural frequency* of the oscillator.

Suppose that we introduce an oscillating driving force $F(t) = mF_0 \cos(\omega t)$, with amplitude (mF_0) and frequency ω. The equation of motion then becomes

$$\frac{d^2y}{dt^2} + \omega_0^2 y = F_0 \cos(\omega t). \tag{3}$$

Suppose that $\omega = \omega_0$, so that the frequency of the driving force exactly matches the natural frequency of the oscillator.

i) Show that the set of solutions of equation (3) is equal to

$$\left\{ \frac{F_0}{2\omega_0} t \sin(\omega_0 t) + b_1 \cos(\omega_0 t) + b_2 \sin(\omega_0 t) : b_1,\ b_2 \in \mathbb{R} \right\}.$$

ii) Suppose that the oscillator starts at rest, so that we have the initial conditions

$$y(0) = 0, \quad \frac{dy}{dt}(0) = 0.$$

Find the solution of equation (3) satisfying these initial conditions and sketch the graph of the solution for $t > 0$.

iii) Suppose that the initial conditions are instead

$$y(0) = 0, \quad \frac{dy}{dt}(0) = 10\omega_0.$$

Find the solution of equation (3) satisfying these initial conditions and sketch the graph of the solution for $t > 0$.

Solution. i) We use the method of undetermined coefficients. Because $\omega = \omega_0$ by hypothesis, we write ω instead of ω_0 throughout for simplicity.

The function $q(t) = F_0 \cos(\omega t)$ is contained in the span of $\{\cos(\omega t),\ \sin(\omega t)\}$, which is the space of solutions of the linear homogeneous equation

$$\left(\frac{d}{dt} - i\omega\right)\left(\frac{d}{dt} + i\omega\right) y = 0.$$

Therefore, we can take

$$p_q\left(\frac{d}{dt}\right) = \left(\frac{d}{dt} - i\omega\right)\left(\frac{d}{dt} + i\omega\right)$$

as our annihilator.

Factoring the operator in the original equation, the equation we are to solve is

$$\left(\frac{d}{dt} - i\omega_0\right)\left(\frac{d}{dt} + i\omega_0\right) y = F_0 \cos(\omega t),$$

and applying the annihilator $p_q\left(\frac{d}{dt}\right)$ to both sides (and remembering that $\omega_0 = \omega$ by hypothesis) converts this to the homogeneous equation

$$\left[\left(\frac{d}{dt} - i\omega\right)\left(\frac{d}{dt} + i\omega\right)\right]^2 y = 0.$$

The basis of solutions of the new equation is

$$\{\cos(\omega t),\ t\cos(\omega t),\ \sin(\omega t),\ t\sin(\omega t)\}.$$

Following the method of undetermined coefficients, we look for a particular solution of the form

$$\phi_p = b_1 t\cos(\omega t) + b_2 t\sin(\omega t)$$

(disregarding the solutions of the associated homogeneous equation for the moment).

Differentiating, we have

$$\begin{aligned}\phi_p' &= b_1(\cos(\omega t) - \omega t\sin(\omega t)) + b_2(\sin(\omega t) + \omega t\cos(\omega t))\\ &= b_1\cos(\omega t) + b_2\sin(\omega t) + \omega b_2 t\cos(\omega t) - \omega b_1 t\sin(\omega t)\end{aligned}$$

and

$$\begin{aligned}\phi_p'' &= -\omega b_1\sin(\omega t) + \omega b_2\cos(\omega t) + \omega b_2\cos(\omega t) - \omega^2 b_2 t\sin(\omega t) - \omega b_1\sin(\omega t) - \omega^2 b_1 t\cos(\omega t)\\ &= 2\omega b_2\cos(\omega t) - 2\omega b_1\sin(\omega t) - \omega^2 b_1 t\cos(\omega t) - \omega^2 b_2 t\sin(\omega t).\end{aligned}$$

We want ϕ_p to satisfy $\phi_p'' + \omega\phi_p = F_0\cos(\omega t)$. Adding,

$$\begin{aligned}&\phi_p'' + \omega^2\phi_p\\ &= 2\omega b_2\cos(\omega t) - 2\omega b_1\sin(\omega t) - \omega^2 b_1 t\cos(\omega t) - \omega^2 b_2 t\sin(\omega t) + \omega^2 b_1 t\cos(\omega t) + \omega^2 b_2 t\sin(\omega t)\\ &= 2\omega b_2\cos(\omega t) - 2\omega b_1\sin(\omega t).\end{aligned}$$

Matching coefficients, we have $b_1 = 0$ and $b_2 = F_0/(2\omega)$.

Therefore,

$$\phi_p = \frac{F_0}{2\omega_0} t\sin(\omega t)$$

and the affine space of all solutions is

$$\frac{F_0}{2\omega_0} t\sin(\omega_0 t) + b_1\cos(\omega_0 t) + b_2\sin(\omega_0 t), \qquad b_1,\ b_2 \in \mathbb{R}.$$

ii) Differentiating the expression for an element of the space of solutions above, we have

$$\frac{F_0}{2\omega_0}\sin(\omega_0 t) + \frac{F_0}{2}t\cos(\omega_0 t) - \omega_0 b_1 \sin(\omega_0 t) + \omega_0 b_2 \cos(\omega_0 t).$$

The initial conditions are $y(0) = 0$, $\frac{dy}{dt}(0) = 0$. These impose the conditions

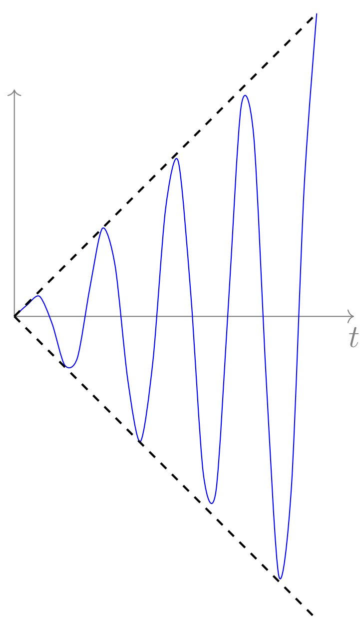
$$\begin{aligned} 0 + b_1 + 0 &= 0 \\ 0 + 0 + 0 + \omega_0 b_2 &= 0, \end{aligned}$$

which implies that $b_1 = b_2 = 0$.

The solution satisfying these initial conditions is

$$\frac{F_0}{2\omega_0}t\sin(\omega_0 t).$$

The solution oscillates with a fixed frequency ω_0, with a linearly growing amplitude!



It is interesting that the oscillating part of the motion is given by $\sin(\omega_0 t)$, so that it is exactly out of phase with the driving force, which is given by $\cos(\omega_0 t)$.

It should be noted that as part of the derivation of the equation of motion of the spring, we made the approximation that the spring obeys Hooke's law. As the spring becomes more and more stretched, Hooke's approximation ceases to hold.

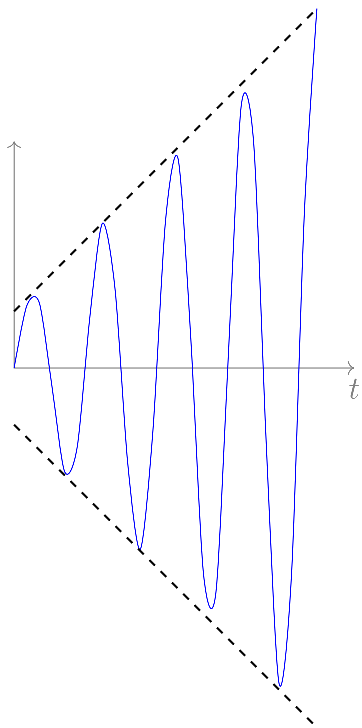
iii) Using the expression for the derivative from the previous part, the initial conditions $y(0) = 0$, $\frac{dy}{dt}(0) = 10\omega_0$ impose the conditions

$$\begin{aligned} 0 + b_1 + 0 &= 0, \\ 0 + 0 + 0 + \omega_0 b_2 &= 10\omega_0. \end{aligned}$$

So that $b_1 = 0,\ b_2 = 10$.

The solution satisfying the initial conditions is

$$\frac{F_0}{2\omega_0} t \sin(\omega_0 t) + 10 \sin(\omega_0 t) = \left(\frac{F_0}{2\omega_0} t + 10 \right) \sin(\omega_0 t).$$



The spring begins already in motion, out of phase with the driving force, and as a result the linear bounds on the amplitude are displaced.

You are encouraged to play around with a few more types of initial condition. For instance, what happens when the initial motion is given by a cosine instead of a sine?

4. Let $c_1, \ldots, c_r$ be a collection of real numbers.

i) Suppose that the numbers $c_1, \ldots, c_r$ are pairwise distinct (meaning $c_i \neq c_j$ if $i \neq j$). Find a linear homogeneous equation whose space of solutions has the set $\{e^{c_1 t}, \ldots, e^{c_r t}\}$ as a basis. Using results from class, conclude that the Wronskian $W(e^{c_1 t}, \ldots, e^{c_r t})(t)$ is not equal to zero for all $t \in \mathbb{R}$.

ii) Conclude that the *Vandermonde determinant* of $c_1, \ldots, c_r$,

$$\det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ c_1 & c_2 & \cdots & c_r \\ c_1^2 & c_2^2 & \cdots & c_r^2 \\ \vdots & \vdots & \ddots & \vdots \\ c_1^{r-1} & c_2^{r-1} & \cdots & c_r^{r-1} \end{pmatrix}$$

is not equal to zero if and only if the numbers $c_1, \dots, c_r$ are pairwise distinct. (*Reminder:* in proving an 'if and only if' statement, there two directions of implication to show. One of the directions here is fairly simple, while the other can be proved using the result of part i).)

The Vandermonde determinant comes up often throughout Mathematics, and the property proved in part ii) is very useful to know.

Solution. i) The equation

$$\left(\frac{d}{dt} - c_1\right)\left(\frac{d}{dt} - c_2\right)\cdots\left(\frac{d}{dt} - c_r\right) y = 0$$

has the desired set as a basis for its space of solutions (with the domain of the solutions being all of $\mathbb{R}$).

Because $\{e^{c_1 t}, \dots, e^{c_r t}\}$ is a basis, it is in particular a linearly independent set. It follows the discussion of the Wronskian in class that there exists a t_0 such that

$$W(e^{c_1 t}, \dots, e^{c_r t})(t_0) \neq 0.$$

Moreover, as discussed in class, since $W(e^{c_1 t}, \dots, e^{c_1 t})(t) = C \exp(-\int a_{r-1}\, dt)$ for some real constant C by Abel's theorem, it follows that if $W(t_0) \neq 0$, then $W(t) \neq 0$ for all t.

Therefore,

$$W(e^{c_1 t}, \dots, e^{c_r t})(t) \neq 0 \qquad \text{for all } t.$$

ii) Suppose first that the numbers $c_1, \dots, c_r$ are pairwise distinct. Computing the Wronskian of $\{e^{c_1 t}, \dots, e^{c_r t}\}$, we have

$$\begin{aligned} W(e^{c_1 t}, \dots, e^{c_r t})(t) &= \det \begin{pmatrix} e^{c_1 t} & e^{c_2 t} & \cdots & e^{c_r t} \\ c_1\, e^{c_1 t} & c_2\, e^{c_2 t} & \cdots & c_r\, e^{c_r t} \\ \vdots & \vdots & \ddots & \vdots \\ c_1^{r-1}\, e^{c_1 t} & c_2^{r-1}\, e^{c_2 t} & \cdots & c_r^{r-1} e^{c_r t} \end{pmatrix} \\ &= e^{c_1 t} e^{c_2 t} \cdots e^{c_r t} \det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ c_1 & c_2 & \cdots & c_r \\ c_1^2 & c_2^2 & \cdots & c_r^2 \\ \vdots & \vdots & \ddots & \vdots \\ c_1^{r-1} & c_2^{r-1} & \cdots & c_r^{r-1} \end{pmatrix} \\ &= \det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ c_1 & c_2 & \cdots & c_r \\ c_1^2 & c_2^2 & \cdots & c_r^2 \\ \vdots & \vdots & \ddots & \vdots \\ c_1^{r-1} & c_2^{r-1} & \cdots & c_r^{r-1} \end{pmatrix} e^{(c_1 + \cdots + c_r)t}. \end{aligned}$$

The coefficient of the exponential is nothing but the Vandermonde determinant!

By part i), the left side does not vanish for all t. Therefore, the right side does not vanish for all t, which would be impossible if the Vandermonde determinant was equal to 0. Therefore, the Vandermonde determinant is not equal to zero.

Remark. Since a_{r-1} (the coefficient of $\frac{d^{r-1}y}{dt^{r-1}}$ in our equation having solutions $\{e^{c_1 t}, \dots, e^{c_r t}\}$) is equal to $-(c_1 + \cdots + c_r)$, Abel's theorem says that

$$W(e^{c_1 t}, \dots, e^{c_r t})(t) = C \exp(-\int a_{r-1}\, dt) = C \exp((c_1 + \cdots + c_r)t)$$

for some constant C. What we found above is that the constant C is exactly the Vandermonde determinant of $c_1, \dots, c_r$!

Conversely, suppose that the Vandermonde determinant of $c_1, \dots, c_r$ is not equal to zero. We would like to show that then $c_1, \dots, c_r$ are pairwise distinct.

We show the contrapositive: if $c_1, \dots, c_r$ are not pairwise distinct, then the Vandermonde determinant of $c_1, \dots, c_r$ is equal to zero

Because $c_1, \dots, c_r$ are not pairwise distinct, at least one pair is equal, say $c_i = c_j$. Then $c_i^k = c_j^k$ for all $k = 0, 1, 2, \dots, r-1$. Therefore, at least two of the columns of the Vandermonde matrix are equal to each other. Hence, the columns of the Vandermonde matrix are linearly dependent, and it follows that the Vandermonde determinant is equal to zero.