1. i) Check that the polynomial differential operators

$$p\left(\frac{d}{dt}\right) = \frac{d^2}{dt^2} + a\frac{d}{dt} + b \qquad \text{and} \qquad q\left(\frac{d}{dt}\right) = \frac{d}{dt} + c, \qquad a, b, c \in \mathbb{C}$$

commute. That is, check that for any function $y\colon I \to \mathbb{C}$,

$$p\left(\frac{d}{dt}\right)\left[q\left(\frac{d}{dt}\right)y\right] = q\left(\frac{d}{dt}\right)\left[p\left(\frac{d}{dt}\right)y\right].$$

Optional Problem. Check that any pair of polynomial differential operators with constant coefficients commutes.

ii) The differential equation

$$\frac{d^4y}{dt^4} - 4\frac{d^3y}{dt^3} + 14\frac{d^2y}{dt^2} - 20\frac{dy}{dt} + 25y = 0 \tag{1}$$

has the characteristic polynomial

$$\chi(z) = z^4 - 4z^3 + 14z^2 - 20z + 25 = \left[(z - (1+2i))(z - (1-2i))\right]^2.$$

Making use of the fact that the corresponding polynomial differential operator factors as

$$\frac{d^4}{dt^4} - 4\frac{d^3}{dt^3} + 14\frac{d^2}{dt^2} - 20\frac{d}{dt} + 25 = \left[\left(\frac{d}{dt} - (1+2i)\right)\left(\frac{d}{dt} - (1-2i)\right)\right]^2,$$

as well as the fact that polynomial differential operators with constant coefficients commute, check that the functions

$$e^{(1+2i)t},\ t\,e^{(1+2i)t},\ e^{(1-2i)t},\ t\,e^{(1-2i)t}$$

are solutions of (1).

Then, taking for granted that complex linear combinations of complex-valued solutions of (1) are again solutions (this follows from the fact that the set of complex-valued solutions of (1) is a complex vector space), show that it follows that

$$e^t\cos(2t),\ t\,e^t\cos(2t),\ e^t\sin(2t),\ t\,e^t\sin(2t)$$

are also solutions of (1).

Remark. The purpose of part ii) is to go through the computations involved in the general proof that functions of the form $t^k e^{wt}$ (with w a root of $\chi(z)$) are solutions of the differential equation $\chi(d/dt)y = 0$ in a specific example. Because we went through the general proof in lecture, it is not necessary to perform this check every time we solve a differential equation; the check is done in this question to help with understanding the general proof.

Solution. i) We have

$$
\begin{aligned}
p\left(\frac{d}{dt}\right)\left[q\left(\frac{d}{dt}\right)y\right] &= p\left(\frac{d}{dt}\right)\left[\left(\frac{d}{dt}+c\right)y\right] \\
&= \left(\frac{d^2}{dt^2}+a\frac{d}{dt}+b\right)\left[\frac{dy}{dt}+cy\right] \\
&= \frac{d^2}{dt^2}\left(\frac{dy}{dt}+cy\right)+a\frac{d}{dt}\left(\frac{dy}{dt}+cy\right)+b\left(\frac{dy}{dt}+cy\right) \\
&= \frac{d^2}{dt^2}\left(\frac{dy}{dt}\right)+\frac{d^2}{dt^2}(cy)+a\frac{d}{dt}\left(\frac{dy}{dt}\right)+a\frac{d}{dt}(cy)+b\frac{dy}{dt}+bcy \\
&= \frac{d^3y}{dt^3}+c\frac{d^2y}{dt^2}+a\frac{d^2y}{dt^2}+ac\frac{dy}{dt}+b\frac{dy}{dt}+bcy \\
&= \frac{d^3y}{dt^3}+(a+c)\frac{d^2y}{dt^2}+(ac+b)\frac{dy}{dt}+bc\,y.
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
q\left(\frac{d}{dt}\right)\left[p\left(\frac{d}{dt}\right)y\right] &= q\left(\frac{d}{dt}\right)\left[\left(\frac{d^2}{dt^2}+a\frac{d}{dt}+b\right)y\right] \\
&= \left(\frac{d}{dt}+c\right)\left[\frac{d^2y}{dt^2}+a\frac{dy}{dt}+by\right] \\
&= \frac{d}{dt}\left(\frac{d^2y}{dt^2}\right)+\frac{d}{dt}\left(a\frac{dy}{dt}\right)+\frac{d}{dt}(by)+c\frac{d^2y}{dt^2}+c\left(a\frac{dy}{dt}\right)+cby \\
&= \frac{d^3y}{dt^3}+a\frac{d^2y}{dt^2}+b\frac{dy}{dt}+c\frac{d^2y}{dt^2}+ac\frac{dy}{dt}+bcy \\
&= \frac{d^3y}{dt^3}+(a+c)\frac{d^2y}{dt^2}+(ac+b)\frac{dy}{dt}+bc\,y.
\end{aligned}
$$

Since both $p\left(\frac{d}{dt}\right)\left[q\left(\frac{d}{dt}\right)y\right]$ and $q\left(\frac{d}{dt}\right)\left[p\left(\frac{d}{dt}\right)y\right]$ are equal to $\frac{d^3y}{dt^3}+(a+c)\frac{d^2y}{dt^2}+(ac+b)\frac{dy}{dt}+bc\,y$, they are equal to each other.

The fact that the coefficients a, b, c are constant is crucial for the above computations.

ii) The function $e^{(1+2i)t}$: We check that

$$\left(\frac{d}{dt}-(1+2i)\right)e^{(1+2i)t}=\frac{d}{dt}e^{(1+2i)t}-(1+2i)e^{(1+2i)t}=(1+2i)e^{(1+2i)t}-(1+2i)e^{(1+2i)t}=0.$$

Therefore,

$$
\begin{aligned}
&\left[\left(\frac{d}{dt}-(1+2i)\right)\left(\frac{d}{dt}-(1-2i)\right)\right]^2 e^{(1+2i)t} \\
&= \left(\frac{d}{dt}-(1-2i)\right)\left(\frac{d}{dt}-(1-2i)\right)\left(\frac{d}{dt}-(1+2i)\right)\left[\left(\frac{d}{dt}-(1+2i)\right)e^{(1+2i)t}\right] \\
&= \left(\frac{d}{dt}-(1-2i)\right)\left(\frac{d}{dt}-(1-2i)\right)\left(\frac{d}{dt}-(1+2i)\right)(0) \\
&= 0,
\end{aligned}
$$

where the last equality is true by linearity of polynomial differential operators. We conclude that the function $e^{(1+2i)t}$ is a solution of the differential equation $\chi(d/dt)y = 0$.

The function $t\,e^{(1+2i)t}$: We check that

$$\begin{aligned}\left(\frac{d}{dt} - (1+2i)\right)(t\,e^{(1+2i)t}) &= \frac{d}{dt}(t\,e^{(1+2i)t}) - (1+2i)t\,e^{(1+2i)t} \\ &= \left(e^{(1+2i)t} + (1+2i)t\,e^{(1+2i)t}\right) - (1+2i)t\,e^{(1+2i)t} \\ &= e^{(1+2i)t}.\end{aligned}$$

Therefore,

$$\left(\frac{d}{dt} - (1+2i)\right)\left[\left(\frac{d}{dt} - (1+2i)\right)(t\,e^{(1+2i)t})\right] = \left(\frac{d}{dt} - (1+2i)\right)\left[e^{(1+2i)t}\right] = 0.$$

We could have reached the same conclusion by applying the lemma shown in class to the effect of

$$\left(\frac{d}{dt} - w\right)^m (f(t)e^{wt}) = \frac{d^m f}{dt^m}e^{wt}, \quad w \in \mathbb{C},\ m \in \mathbb{N}.$$

Taking $w = 1+2i$, $f = t$ and $m = 2$, we have

$$\left(\frac{d}{dt} - (1+2i)\right)^2 (t\,e^{(1+2i)t}) = \frac{d^2(t)}{dt^2}e^{(1+2i)t} = 0\,e^{(1+2i)t} = 0.$$

Thus,

$$\begin{aligned}&\left[\left(\frac{d}{dt} - (1+2i)\right)\left(\frac{d}{dt} - (1-2i)\right)\right]^2 (t\,e^{(1+2i)t}) \\ &= \left(\frac{d}{dt} - (1-2i)\right)\left(\frac{d}{dt} - (1-2i)\right)\left[\left(\frac{d}{dt} - (1+2i)\right)^2 (t\,e^{(1+2i)t})\right] \\ &= \left(\frac{d}{dt} - (1-2i)\right)\left(\frac{d}{dt} - (1-2i)\right)(0) \\ &= 0,\end{aligned}$$

and so the function $t\,e^{(1+2i)t}$ is a solution of the differential equation $\chi(d/dt)y = 0$.

The other two functions are checked similarly: we can check that $e^{(1-2i)t}$ is annihilated by $(\frac{d}{dt} - (1-2i))$ and $t\,e^{(1-2i)t}$ is annihilated by $(\frac{d}{dt} - (1-2i))^2$ —

$$\left(\frac{d}{dt} - (1-2i)\right)e^{(1-2i)t} = \frac{d}{dt}e^{(1-2i)t} - (1-2i)e^{(1-2i)t} = (1-2i)e^{(1-2i)t} - (1-2i)e^{(1-2i)t} = 0$$

and

$$\begin{aligned}\left(\frac{d}{dt} - (1-2i)\right)(t\,e^{(1-2i)t}) &= \frac{d}{dt}(t\,e^{(1-2i)t}) - (1-2i)t\,e^{(1-2i)t} \\ &= \left(e^{(1-2i)t} + (1-2i)t\,e^{(1-2i)t}\right) - (1-2i)t\,e^{(1-2i)t} \\ &= e^{(1-2i)t},\end{aligned}$$

so

$$\left(\frac{d}{dt}-(1-2i)\right)\left[\left(\frac{d}{dt}-(1-2i)\right)(t\,e^{(1-2i)t})\right]=\left(\frac{d}{dt}-(1-2i)\right)\left[e^{(1-2i)t}\right]=0.$$

Therefore, both $e^{(1-2i)t}$ and $t\,e^{(1-2i)t}$ are solutions of $\chi(d/dt)y=0$ —

$$\begin{aligned}&\left[\left(\frac{d}{dt}-(1+2i)\right)\left(\frac{d}{dt}-(1-2i)\right)\right]^2 e^{(1+2i)t}\\&=\left(\frac{d}{dt}-(1+2i)\right)\left(\frac{d}{dt}-(1+2i)\right)\left(\frac{d}{dt}-(1-2i)\right)\left[\left(\frac{d}{dt}-(1-2i)\right)e^{(1-2i)t}\right]\\&=\left(\frac{d}{dt}-(1+2i)\right)\left(\frac{d}{dt}-(1+2i)\right)\left(\frac{d}{dt}-(1-2i)\right)(0)\\&=0,\end{aligned}$$

and

$$\begin{aligned}&\left[\left(\frac{d}{dt}-(1+2i)\right)\left(\frac{d}{dt}-(1-2i)\right)\right]^2 (t\,e^{(1+2i)t})\\&=\left(\frac{d}{dt}-(1+2i)\right)\left(\frac{d}{dt}-(1+2i)\right)\left[\left(\frac{d}{dt}-(1-2i)\right)^2(t\,e^{(1-2i)t})\right]\\&=\left(\frac{d}{dt}-(1+2i)\right)\left(\frac{d}{dt}-(1+2i)\right)(0)\\&=0.\end{aligned}$$

Finally, we have

$$\begin{aligned}\frac{1}{2}e^{(1+2i)t}+\frac{1}{2}e^{(1-2i)t}&=\frac{e^te^{2it}+e^te^{-2it}}{2}=e^t\left(\frac{e^{2it}+e^{-2it}}{2}\right)=e^t\cos(2t),\\ \frac{1}{2i}e^{(1+2i)t}-\frac{1}{2i}e^{(1-2i)t}&=\frac{e^te^{2it}-e^te^{-2it}}{2i}=e^t\left(\frac{e^{2it}-e^{-2it}}{2i}\right)=e^t\sin(2t),\\ \frac{1}{2}te^{(1+2i)t}+\frac{1}{2}te^{(1-2i)t}&=t\frac{e^te^{2it}+e^te^{-2it}}{2}=te^t\left(\frac{e^{2it}+e^{-2it}}{2}\right)=te^t\cos(2t),\\ \frac{1}{2i}te^{(1+2i)t}-\frac{1}{2i}te^{(1-2i)t}&=t\frac{e^te^{2it}-e^te^{-2it}}{2i}=te^t\left(\frac{e^{2it}-e^{-2it}}{2i}\right)=te^t\sin(2t).\end{aligned}$$

Since linear combinations of solutions are again solutions, we conclude that

$$e^t\cos(2t),\ t\,e^t\cos(2t),\ e^t\sin(2t),\ t\,e^t\sin(2t)$$

are also solutions of (1).

2. In this problem, we look at a cylindrical cork bobbing up and down in a pool of fluid.

Let d denote the density of the cork, and let ρ denote the density of the fluid (both are assumed uniform). Assume that $d<\rho$. Let A denote the area of the circular face of the cork, and let ℓ denote the length of the cork (measured perpendicular to the circular face).

Choose coordinates so that the surface of the fluid is in the xy-plane, and the positive z-direction points out of the pool of fluid. Assume that gravity is uniform, with gravitational constant g, and points in the negative z-direction.

i) By Archimedes' principle, the fluid exerts a buoyant force on the cork (in the positive z direction) equal to the weight (= mass times gravitational constant) of the fluid displaced by the cork.

   If the cork floats at rest in the pool of fluid, the weight of the fluid displaced by the submerged part of the cork must be equal to the total weight of the cork.

   Suppose that the cork floats at rest with its circular face parallel to the surface of the fluid. Let h denote the height of the submerged part of the cork measured from its bottom face. Show that

$$h = \left(\frac{d}{\rho}\right)\ell.$$

ii) Let z denote the displacement of the cork from the rest level h (take the displacement to be positive along the positive z direction, so that the cork is still oriented with its circular faces parallel to the surface of the fluid, and small enough so that the cork does not completely leave the fluid). Show that the sum of the gravitational force on the cork and the buoyant force is

$$-A\rho g z\ \mathbf{e_z},$$

   where $\mathbf{e_z}$ is the unit vector in the positive z-direction.

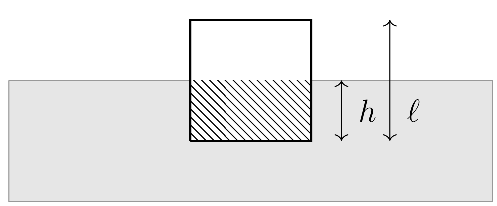
iii) As long as the motion of the cork is not too rapid (and the fluid is sufficiently viscous), the fluid exerts a drag force on the cork proportional to the velocity of its motion, and directed opposite to the velocity. (This is called the laminar case. If there is turbulence, the drag is proportional to the square of the velocity, and the problem is no longer linear.) Thus, the net force on the cork is $\left(-b\frac{dz}{dt} - A\rho g z\right)\mathbf{e_z}$. By Newton's second law, the equation of motion is

$$\underbrace{A\ell d}_{\text{Mass of cork}}\frac{d^2z}{dt^2} + b\frac{dz}{dt} + A\rho g z = 0.$$

   In terms of the constants A, ℓ, d, b, ρ, g, characterize when the resulting motion will be overdamped, critically damped, and underdamped. Briefly describe the motion of the cork in each of these three cases.

Solution. i) In these solutions, we refer to the fluid simply as water (though it could also be, for example, honey, vinegar, liquid helium, wet cement, and so on).

A vertical cross-section of the situation looks as follows:



The shaded part of the cork is underwater.

The volume of the water displaced by the cork is equal to $h\,A$, hence the mass of the water displaced by the cork is $(h\,A)\rho$, and the weight of the water displaced by the cork is $(h\,A)\rho g$. The weight of the cork is similarly computed to be $(\ell A)dg$.

When the cork floats at rest, the gravity on the cork is balanced by the buoyant force, and we have

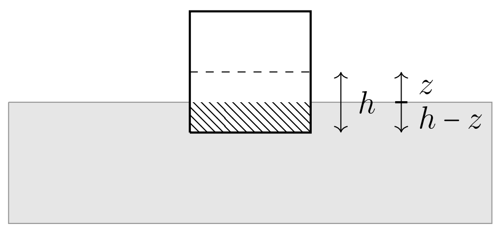
$$hA\rho g = \ell Adg.$$

The constants A and g cancel out, and, solving for h, we find

$$h = \left(\frac{d}{\rho}\right)\ell.$$

We can interpret this result as follows: the proportion of the cork that is underwater is equal to the ratio of the two densities d/ρ. Note that when $d > \rho$, that is the cork is denser than the fluid, we get the absurd result that $h > \ell$. In this case, the cork will sink!

ii) The length of the cork that is underwater will be equal to $h - z$:



Therefore, the buoyant force is equal to $((h-z)A\rho g)\,\mathbf{e_z}$. The gravitational force on the cork is still $-(\ell Adg)\,\mathbf{e_z}$.

The sum of the two forces is

$$((h-z)A\rho g - \ell Adg)\ \mathbf{e_z} = (hA\rho g - \ell Adg - zA\rho g)\ \mathbf{e_z}.$$

Since $hA\rho g = \ell Adg$, the first two terms cancel out and we get

$$-A\rho g z\,\mathbf{e_z}.$$

iii) Dividing through by the mass $A\ell d$, we get a second order linear homogeneous equation in standard form:

$$\frac{d^2z}{dt^2} + \frac{b}{A\ell d}\frac{dz}{dt} + \frac{\rho g}{\ell d}z = 0. \tag{2}$$

The behaviour of the solutions depends on the sign of the discriminant

$$\left(\frac{b}{A\ell d}\right)^2 - 4\frac{\rho g}{\ell d} = \frac{b^2 - 4A^2\ell d\rho g}{(A\ell d)^2} = \frac{b^2 - 4Am\rho g}{m^2},$$

of the characteristic polynomial of (2) (we have introduced the notation $m = A\ell d$ for the mass of the cork).

Now, the sign of the discriminant above is the same as the sign of its numerator $b^2 - 4Am\rho g$.

There are three cases: $b^2 > 4Am\rho g$, $b^2 = 4Am\rho g$ and $b^2 < 4Am\rho g$.

Because $b > 0$ by (unstated) assumption, we can take the positive square root in each case without loss of generality. The three cases are

- $b > 2\sqrt{Am\rho g}$: The characteristic polynomial has two distinct real roots. The motion is *overdamped*. One can imagine the cork slowly sinking into very dense fluid;
- $b = 2\sqrt{Am\rho g}$: The characteristic polynomial has a repeated real root. The motion is *critically damped*. The cork returns to its rest state, possibly bobbing at most once;
- $b < 2\sqrt{Am\rho g}$: The characteristic polynomial has two conjugate complex roots. The motion is *underdamped*. The cork oscillates up and down, with an exponentially decreasing amplitude.

3. Using Abel's theorem, find the Wronskian of two solutions of Bessel's equation

$$\frac{d^2y}{dt^2} + \frac{1}{t}\frac{dy}{dt} + \left(1 - \frac{\nu^2}{t^2}\right)y = 0, \quad t > 0,$$

up to a real constant.

Solution. Abel's theorem says that if $\phi_1, \dots, \phi_r$ is a set of solutions (not necessarily linearly independent) of the linear homogeneous differential equation

$$\frac{d^r y}{dt^r} + a_{r-1}(t)\frac{d^{r-1}y}{dt^{r-1}} + \cdots + a_0(t)y = 0,$$

over I, where $a_{r-1}, \dots, a_0$ are continuous over I, then there exists a real number $C \in \mathbb{R}$ such that

$$W(\phi_1, \dots, \phi_r)(t) = C\exp\left(-\int a_{r-1}(t)\,dt\right).$$

For Bessel's equation, $a_{r-1}(t) = a_1(t) = t^{-1}$. We have

$$-\int a_1(t)\,dt = -\int \frac{dt}{t} = -\ln(t) = \ln\left(\frac{1}{t}\right),$$

and so

$$W(\phi_1, \phi_2)(t) = C\exp(\ln(1/t)) = \frac{C}{t},$$

for some real number C.