MTHE 237 — PROBLEM SET 04 SOLUTIONS

1. Solve the following differential equations.

i) $\dfrac{dy}{dt} + 2017y = 0, \quad y(0) = 5.$

ii) $\dfrac{d^2y}{dt^2} - 10\dfrac{dy}{dt} + 21y = 0, \quad y(0) = 5, \quad \dfrac{dy}{dt}(0) = 19.$

iii) $\dfrac{d^2y}{dt^2} - 6\dfrac{dy}{dt} + 25y = 0, \quad y(0) = 1, \quad \dfrac{dy}{dt}(0) = 15.$

iv) $\dfrac{d^2y}{dt^2} + 8\dfrac{dy}{dt} + 16y = 0, \quad y(0) = 1, \quad \dfrac{dy}{dt}(1) = 10.$

v) $\dfrac{d^6y}{dt^6} - y = 0.$ (It is sufficient to find a basis for the space of solutions.)

Solution. i) The characteristic polynomial is $\chi(z) = z + 2017$. This has a single real root at $z = -2017$. Therefore, a basis of solutions is

$$\{e^{-2017t}\}.$$

Elements of the solution space have the form

$$c\,e^{-2017t}, \quad c \in \mathbb{R}.$$

Imposing the initial condition $y(0) = 5$, we find

$$c\,e^{0} = 5 \quad \text{so } c = 5.$$

The solution is

$$5e^{-2017t}, \quad t \in \mathbb{R}.$$

ii) The characteristic polynomial is $\chi(z) = z^2 - 10z + 21 = (z-3)(z-7)$. This has two distinct real roots. Therefore, a basis of solutions is

$$\{e^{3t}, e^{7t}\}.$$

Elements of the solution space have the form

$$c_1e^{3t} + c_2e^{7t}, \quad c_1, c_2 \in \mathbb{R}.$$

Imposing the initial conditions,

$$\begin{aligned} c_1e^0 + c_2e^0 &= 5, \\ 3c_1e^0 + 7c_2e^0 &= 19. \end{aligned}$$

So that

$$\begin{aligned} c_1 + c_2 &= 5, \\ 3c_1 + 7c_2 &= 19. \end{aligned}$$

Subtracting three times the first equation from the second, we obtain $4c_2 = 4$, so $c_2 = 1$, and so, from the first equation, $c_1 = 4$.

The solution is

$$4e^{3t} + e^{7t}, \quad t \in \mathbb{R}.$$

iii) The characteristic polynomial is $\chi(z) = z^2 - 6z + 25$. Using the quadratic formula, we find that the roots are

$$\frac{6 \pm \sqrt{6^2 - 4 \cdot 25}}{2} = \frac{6 \pm \sqrt{-64}}{2} = 3 \pm 4i.$$

A basis of solutions is

$$\{e^{3t}\cos(4t),\, e^{3t}\sin(4t)\}.$$

Elements of the space of solutions look like

$$c_1 e^{3t}\cos(4t) + c_2 e^{3t}\sin(4t) = e^{3t}(c_1\cos(4t) + c_2\sin(4t)), \quad c_1, c_2 \in \mathbb{R}.$$

The derivative of such a solution is

$$3e^{3t}(c_1\cos(4t)+c_2\sin(4t))+e^{3t}(-4c_1\sin(4t)+4c_2\cos(4t)) = e^{3t}((3c_1+4c_2)\cos(4t)+(3c_2-4c_1)\sin(4t)).$$

Imposing the initial conditions,

$$\begin{aligned} e^0(c_1\cos(0) + c_2\sin(0)) &= 1, \\ e^0((3c_1 + 4c_2)\cos(0) + (3c_2 - 4c_1)\sin(0)) &= 15, \end{aligned}$$

so that

$$\begin{aligned} c_1 &= 1, \\ 3c_1 + 4c_2 &= 15. \end{aligned}$$

Therefore, $c_1 = 1$ and $4c_2 = 12$, so that $c_2 = 3$.

The solution is

$$e^{3t}\left(\cos(4t) + 3\sin(4t)\right), \quad t \in \mathbb{R}.$$

iv) The characteristic polynomial is $\chi(z) = z^2 + 8z + 16 = (z + 4)^2$. This has a real double root at $z = -4$. Therefore, a basis of solutions is

$$\{e^{-4t},\ te^{-4t}\}.$$

The elements of the space of solutions look like

$$c_1 e^{-4t} + c_2 t e^{-4t}, \quad c_1, c_2 \in \mathbb{R}.$$

The derivative of a solution is

$$-4c_1 e^{-4t} + c_2(e^{-4t} - 4te^{-4t}) = (-4c_1 + (1 - 4t)c_2)e^{-4t}.$$

Imposing the initial conditions,

$$c_1 e^0 + c_2 \cdot 0 = 1,$$
$$(-4c_1 + (1-4)c_2)e^{-4} = 10.$$

Therefore, $c_1 = 1$ and $c_2 = -(10e^4 + 4)/3$. The solution is

$$e^{-4t} - \frac{10e^4 + 4}{3} t e^{-4t}, \quad t \in \mathbb{R}.$$

v) The characteristic polynomial $\chi(z) = z^6 - 1$. The roots are the sixth roots of unity, which are, as discussed in lecture, of the form $\exp\left(i\frac{2\pi k}{6}\right)$, $k = 0, 1, \ldots, 5$.

Writing out the sixth roots of unity,

$$\exp(0), \ \exp\left(i\frac{2\pi}{6}\right), \ \exp\left(i\frac{4\pi}{6}\right), \ \exp\left(i\frac{6\pi}{6}\right), \ \exp\left(i\frac{8\pi}{6}\right), \ \exp\left(i\frac{10\pi}{6}\right).$$

These expressions simplify to

$$1, \ \exp\left(i\frac{\pi}{3}\right), \ \exp\left(i\frac{2\pi}{3}\right), \ \exp(i\pi) = -1, \ \exp\left(i\frac{4\pi}{3}\right), \ \exp\left(i\frac{5\pi}{3}\right).$$

By Euler's formula, these are equal to (using the facts that $\cos(\pi/3) = 1/2$ and $\sin(\pi/3) = \sqrt{3}/2$)

$$1, \ \frac{1+i\sqrt{3}}{2}, \ \frac{-1+i\sqrt{3}}{2}, \ -1, \ \frac{-1-i\sqrt{3}}{2}, \ \frac{1-i\sqrt{3}}{2}.$$

There are two distinct real roots at $z = 1$ and $z = -1$, as well as two conjugate pairs of complex roots. A basis for the space of solutions is given by

$$\left\{ e^t, \ e^{-t}, \ e^{t/2}\cos\left(\frac{\sqrt{3}}{2}t\right), \ e^{t/2}\sin\left(\frac{\sqrt{3}}{2}t\right), \ e^{-t/2}\cos\left(\frac{\sqrt{3}}{2}t\right), \ e^{-t/2}\sin\left(\frac{\sqrt{3}}{2}t\right) \right\}.$$

2. In each of the following, find a linear homogeneous differential equation with constant coefficients with the given functions as a basis for its space of solutions.

i) $\phi_1(t) = e^t, \quad \phi_2(t) = e^{2t}.$

ii) $\phi_1(t) = e^t, \quad \phi_2(t) = e^{2t}, \quad \phi_3(t) = e^{3t}.$

iii) $\phi_1(t) = e^{-kt}, \quad \phi_2(t) = t\,e^{-kt}, \quad \phi_3(t) = e^{2t},$ where k is a real number.

iv) $\phi_1(t) = e^{\sigma t}\cos(\omega t), \quad \phi_2(t) = e^{\sigma t}\sin(\omega t),$ where σ and $\omega \neq 0$ are real numbers.

v) $\phi_1(t) = 1, \quad \phi_2(t) = t, \quad \phi_3(t) = t^2, \quad \phi_4(t) = e^{-t}\cos(t), \quad \phi_5(t) = e^{-t}\sin(t).$

Solution. i) The characteristic polynomial should have two distinct real roots at $z = 1$ and $z = 2$, so should be equal to

$$\chi(z) = (z-1)(z-2) = z^2 - 3z + 2.$$

The corresponding differential equation is

$$\frac{d^2y}{dt^2} - 3\frac{dy}{dt} + 2y = 0.$$

ii) The characteristic polynomial should have three distinct real roots at $z = 1, z = 2$ and $z = 3$, so should be equal to

$$\chi(z) = (z-1)(z-2)(z-3) = (z^2 - 3z + 2)(z-3) = z^3 - 6z^2 + 11z - 6.$$

The corresponding differential equation is

$$\frac{d^3y}{dt^3} - 6\frac{d^2y}{dt^2} + 11\frac{dy}{dt} - 6y = 0.$$

iii) The characteristic polynomial should have a double root at $z = -k$, and a single root at $z = 2$. So, it should be equal to

$$\chi(z) = (z+k)^2(z-2) = (z^2 + 2kz + k^2)(z-2) = z^3 + 2(k-1)z^2 + k(k-4)z - 2k^2.$$

The corresponding differential equation is

$$\frac{d^3y}{dt^3} + 2(k-1)\frac{d^2y}{dt^2} + k(k-4)\frac{dy}{dt} - 2k^2y = 0.$$

iv) The characteristic polynomial should have roots at $z = \sigma + i\omega$ and $z = \sigma - i\omega$, so should be equal to

$$\chi(z) = (z - (\sigma + i\omega))(z - (\sigma - i\omega)) = z^2 - 2\sigma z + (\sigma^2 + \omega^2).$$

The corresponding differential equation is

$$\frac{d^2y}{dt^2} - 2\sigma\frac{dy}{dt} + (\sigma^2 + \omega^2)y = 0.$$

v) The characteristic polynomial should have a triple root at $z = 0$, and a pair of complex roots $z = -1 \pm i$.

$$\chi(z) = z^3(z - (-1+i))(z - (-1-i)) = z^3(z^2 + 2z + 2) = z^5 + 2z^4 + 2z^3.$$

The corresponding differential equation is

$$\frac{d^5y}{dt^5} + 2\frac{d^4y}{dt^4} + 2\frac{d^3y}{dt^3} = 0.$$

3. By comparing the real and imaginary parts of the identity

$$e^{i(\theta+\phi)} = e^{i\theta}e^{i\phi},$$

show the angle addition identities for sin and cos:

$$\begin{aligned}\sin(\theta+\phi) &= \sin(\theta)\cos(\phi)+\cos(\theta)\sin(\phi), \quad \text{and}\\ \cos(\theta+\phi) &= \cos(\theta)\cos(\phi)-\sin(\theta)\sin(\phi).\end{aligned}$$

Solution. By Euler's formula, the left-hand side is

$$e^{i(\theta+\phi)} = \cos(\theta+\phi)+i\sin(\theta+\phi).$$

The right-hand side is

$$(\cos(\theta)+i\sin(\theta))(\cos(\phi)+i\sin(\phi)) = \cos(\theta)\cos(\phi)+i\cos(\theta)\sin(\phi)+i\sin(\theta)\cos(\phi)-\sin(\theta)\sin(\phi).$$

Comparing the real and imaginary parts of the two sides, we see that

$$\cos(\theta+\phi) = \text{real part of } e^{i(\theta+\phi)} = \cos(\theta)\cos(\phi)-\sin(\theta)\sin(\phi)$$

and

$$\sin(\theta+\phi) = \text{imaginary part of } e^{i(\theta+\phi)} = \cos(\theta)\sin(\phi)+\sin(\theta)\cos(\phi).$$

4 (Simple harmonic motion)**.** Consider a mass hanging on a spring with spring constant k. After an initial stretch of the spring to balance the force of gravity, the mass will hang at rest. Choose a coordinate system such that the y-axis is aligned with the spring, and such that the rest point of the mass is at $y=0$.

If the mass is moved a distance y from $y=0$, it will be acted on by a restoring force due do the spring, given by Hooke's law: $\mathbf{F}_{\text{restoring}} = -ky$. In the absence of other forces (such as damping), the motion of the mass is described by

$$m\frac{d^2y}{dt^2} = -ky \qquad \text{(Newton's second law)},$$

or, bringing to standard form for a linear equation,

$$\frac{d^2y}{dt^2}+\omega^2 y = 0, \qquad \text{where } \omega^2 = k/m. \tag{1}$$

i) Find the roots of the characteristic polynomial of (1), and conclude that $\phi_1(t) = \cos(\omega t)$ and $\phi_2(t) = \sin(\omega t)$ are a basis for the space of solutions of (1).

ii) Check that for real numbers $A \geq 0$ and $\phi \in (-\pi,\pi]$, the function

$$A\cos(\omega t+\phi),$$

is a solution of (1). Therefore, we have

$$A\cos(\omega t+\phi) = c_1\cos(\omega t)+c_2\sin(\omega t)$$

for some real numbers c_1, c_2. Find c_1 and c_2 in terms of A and ϕ. (*Suggestion:* Expand $\cos(\omega t+\phi)$ using the angle addition identity, and make use of the fact that any vector is expressed uniquely as a linear combination of basis elements.)

iii) Find A and ϕ so that the function $A\cos(\omega t+\phi)$ is a solution of (1) with initial conditions

$$y(0) = y_0, \qquad \frac{dy}{dt}(0) = v_0\omega.$$

Conclude that any solution of (1) may be written in the form $A\cos(\omega t + \phi)$.

iv) The parameters ω, A and ϕ are called the frequency, amplitude and phase of the motion, respectively. Briefly describe (with sketches, if possible) how the graph of $A\cos(\omega t+\phi)$ depends on these three parameters.

Solution. i) The characteristic polynomial of (1) is $\chi(z) = z^2 + \omega^2 = (z + i\omega)(z - i\omega)$. This has two complex roots at $\pm i\omega$, so by the general procedure for solving linear homogeneous differential equations with constant coefficients the two claimed functions are a basis of the space of solutions.

ii) Differentiating twice, we find

$$\frac{d}{dt}A\cos(\omega t + \phi) = -A\omega\sin(\omega t + \phi)$$
$$\frac{d^2}{dt^2}A\cos(\omega t + \phi) = -A\omega^2\cos(\omega t + \phi).$$

Therefore,

$$\frac{d^2 A\cos(\omega t + \phi)}{dt^2} + \omega^2 A\cos(\omega t + \phi) = 0,$$

so that $A\cos(\omega t + \phi)$ is a solution of equation (1).

Expanding $\cos(\omega t + \phi)$ using the cosine angle-addition identity from question 3, we have

$$\cos(\omega t + \phi) = \cos(\omega t)\cos(\phi) - \sin(\omega t)\sin(\phi).$$

Therefore,

$$A\cos(\omega t + \phi) = A\cos(\phi)\cos(\omega t) - A\sin(\phi)\sin(\omega t),$$

so that

$$c_1 = A\cos(\phi) \quad \text{and} \quad c_2 = -A\sin(\phi).$$

iii) Differentiating, we have

$$\frac{d}{dt}A\cos(\omega t + \phi) = -A\omega\sin(\omega t + \phi).$$

Therefore, we are looking for A and ϕ so that

$$y_0 = y(0) = A\cos(0 + \phi) = A\cos(\phi),$$
$$v_0\omega = \frac{dy}{dt}(0) = -A\omega\sin(\phi) \qquad \text{so that} \qquad v_0 = -A\sin(\phi).$$

(The last computation assumes that $\omega \neq 0$.) Adding the squares of the two initial conditions, we find

$$y_0^2 + v_0^2 = A^2\cos^2(\phi) + A^2\sin^2(\phi) = A^2,$$

so that necessarily

$$A = \sqrt{y_0^2 + v_0^2}.$$

Dividing the two solutions, we find

$$-\frac{v_0}{y_0} = -\frac{-A\sin(\phi)}{A\cos(\phi)} = \tan(\phi).$$

Now, solving the last equation for ϕ is a little subtle. The solution depends on the signs of y_0 and v_0.

The solution is

$$\phi = \begin{cases} \arctan\left(-\frac{v_0}{y_0}\right) + \frac{\pi}{2}, & y_0 > 0,\ -v_0 > 0 \\ \frac{\pi}{2}, & y_0 = 0,\ -v_0 > 0 \\ \arctan\left(-\frac{v_0}{y_0}\right), & y_0 > 0,\ -v_0 > 0 \\ -\frac{\pi}{2}, & y_0 = 0,\ -v_0 < 0 \\ \arctan\left(-\frac{v_0}{y_0}\right) - \frac{\pi}{2}, & y_0 < 0,\ -v_0 < 0 \end{cases}.$$

With these choices of A and ϕ, the function $A\cos(\omega t + \phi)$ satisfies the initial conditions

$$y(0) = y_0, \qquad \frac{dy}{dt}(0) = v_0\omega.$$

By part ii), the functions $A\cos(\omega t+\phi)$ are solutions for any A and ϕ, and so we obtain a map

$$\{\text{Functions of form } A\cos(\omega t + \phi)\} \longrightarrow \left\{\text{Solutions of } \frac{d^2y}{dt^2} + \omega^2 y = 0\right\}$$

Because any initial condition may be realized by some choice of A and ϕ, it follows that this map is surjective.

(The vector space of solutions is isomorphic to $\mathbb{R}^2$, with one isomorphism given by

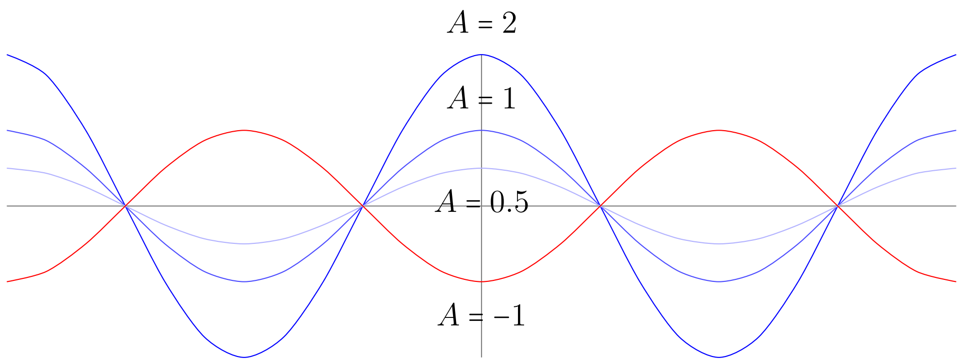
$$\Psi_0 : \phi \mapsto \left(\phi(0), \frac{d\phi}{dt}(0)\right),$$

where ϕ is a solution of $d^2y/dt^2 + \omega^2 y = 0$. So, if any initial condition may be realized by a function of the form $A\cos(\omega t + \phi)$, it follows that the composition

$$\{\text{Functions of form } A\cos(\omega t + \phi)\} \longrightarrow \left\{\text{Solutions } \phi \text{ of } \frac{d^2y}{dt^2} + \omega^2 y = 0\right\} \longrightarrow \left\{\left(\phi(0), \frac{d\phi}{dt}(0)\right)\right\}$$

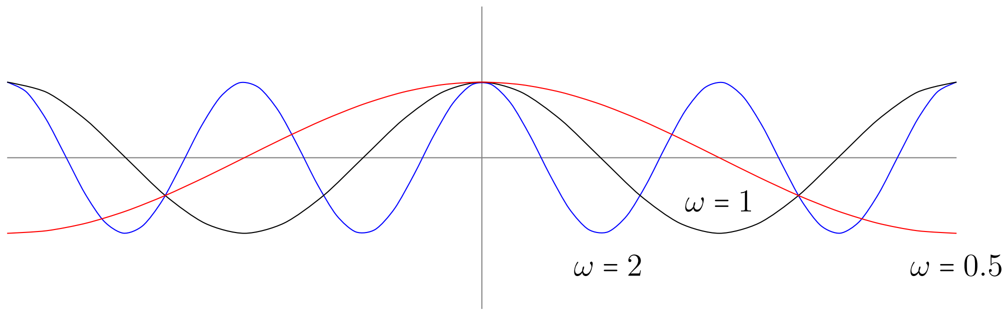
is surjective. It then follows that the first map is also surjective.)

iv) The amplitude A scales the graph of $A\cos(\omega t + \phi)$ vertically.

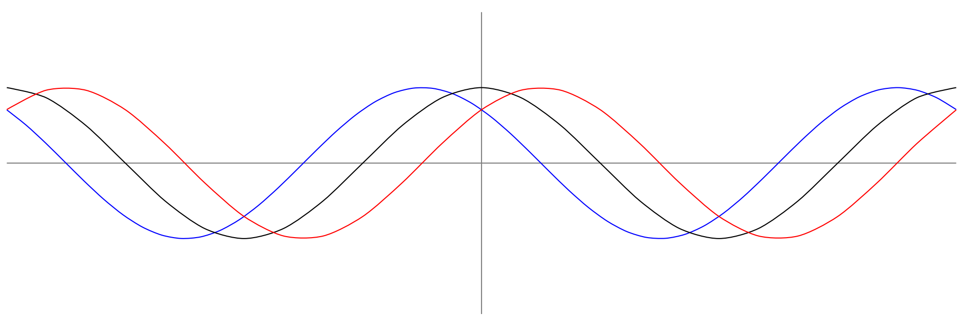


In terms of the simple harmonic oscillator, the y-coordinate of the motion oscillates between $+A$ and $-A$.

The frequency changes how often the graph of $A\cos(\omega t + \phi)$ oscillates.



The phase shifts the graph of $A\cos(\omega t + \phi)$ horizontally.



The blue graph is has $\phi = \pi/4$ (the graph is shifted left) and the red graph has $\phi = -\pi/4$ (the graph is shifted right).