Mthe 237 — Problem Set 03 Solutions

1. i) Check that the differential equation $(y^2+y)-x\frac{dy}{dx}=0$ is not exact. Now, multiplying both sides by $\mu(x,y)=y^{-2}$ yields the differential equation $\left(1+\frac{1}{y}\right)-\frac{x}{y^2}\frac{dy}{dx}=0$. Check that the latter is exact (for $y\neq 0$).

If a differential equation that is not exact is converted to one that is exact by multiplying both sides of the equation by a function $\mu(x,y)$, as in the example of i), the function $\mu(x,y)$ is called an *integrating factor*.

Solutions of the equation multiplied by μ are also solutions of the original equation, as long as we avoid regions where μ is zero.

Although in principle an integrating factor exists for every equation of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0, \tag{1}$$

in practice an integrating factor is frequently hard to find, and there is no known general method for finding it.

One approach to trying to find an integrating factor is to guess that it has a special simple form, and try your luck. For example, suppose we *guessed* that the differential equation (1) has an integrating factor $\mu(x)$ that is a function of x only, so that

$$\mu(x)M(x,y)+\mu(x)N(x,y)\frac{dy}{dx}=0$$

is now exact.

ii) Show that the exactness condition

$$\frac{\partial(\mu(x)M(x,y))}{\partial y}=\frac{\partial(\mu(x)N(x,y))}{\partial x}$$

may be rearranged to

$$\frac{1}{\mu}\frac{d\mu}{dx}=\frac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{N}. \tag{2}$$

If the right side of eq. (2) is a function of x only, then eq. (2) is separated equation for $\mu(x)$, which we can solve by the usual method for separated equations (integrating both sides).

iii) The equation $(e^x-\sin(y))+\cos(y)\frac{dy}{dx}=0$ is not exact, but has an integrating factor that may be found by the above method. Find it.

iv) Check that for the equation $xy^2+x\frac{dy}{dx}=0$, the expression $\dfrac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{N}$ is not a function of x only. Therefore, the above method does not yield an integrating factor.

Other common guesses for the form of an integrating factor are $\mu(y)$, $\mu(xy)$, $\mu(x/y)$, $\mu(y/x)$. Each, in lucky cases, leads to a separable differential equation for μ.[1] However, all five may

[1] For instance, $\mu(xy)=1/(xy(1-xy))$ works for the equation of part iv).

fail to yield an integrating factor. Because there is no known general method of finding integrating factors, their usefulness is limited.

The idea of integrating factors does lead to a general method of solving first-order linear equations, however, which we shall cover in detail in lecture!

Solution. i) We compute that for the differential equation $(y^2+y) - x\,\frac{dy}{dx} = 0$,

$$\frac{\partial M}{\partial y} = 2y+1, \quad \frac{\partial N}{\partial x} = -1.$$

Because the two partials are not equal in any open rectangle, the equation $(y^2+y) - x\,\frac{dy}{dx} = 0$ is not exact.

On the other hand, for $\left(1+\frac{1}{y}\right) - \frac{x}{y^2}\,\frac{dy}{dx} = 0$, we have

$$\frac{\partial M}{\partial y} = -\frac{1}{y^2}, \quad \frac{\partial N}{\partial x} = -\frac{1}{y^2}.$$

The two partials are equal for $y > 0$ and $y < 0$. We conclude that on either of these two open rectangles the equation is exact .

ii) We have

$$\begin{aligned}\frac{\partial\left(\mu(x)M\right)}{\partial y} &= \mu\frac{\partial M}{\partial y},\\ \frac{\partial\left(\mu(x)N\right)}{\partial x} &= \frac{d\mu}{dx}N + \mu\,\frac{\partial N}{\partial x}.\end{aligned}$$

Setting the two partials equal to each other,

$$\begin{aligned}\mu\frac{\partial M}{\partial y} &= \frac{d\mu}{dx}N + \mu\,\frac{\partial N}{\partial x}, \quad \text{so that}\\ \frac{d\mu}{dx}N &= \mu\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right), \quad \text{and}\\ \frac{1}{\mu}\,\frac{d\mu}{dx} &= \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}.\end{aligned}$$

iii) For the equation $(e^x - \sin(y)) + \cos(y)\,\frac{dy}{dx} = 0$, we have

$$\begin{aligned}\frac{\partial M}{\partial y} &= -\cos(y),\\ \frac{\partial N}{\partial x} &= 0,\\ N &= \cos(y), \quad \text{and}\\ \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} &= \frac{-\cos(y) - 0}{\cos(y)} = -1.\end{aligned}$$

The expression is constant with x and y, and so in particular a function of x only.

The separated equation for μ is

$$\frac{1}{\mu}\frac{d\mu}{dx} = -1. \tag{3}$$

Integrating both sides, solutions are given implicitly by

$$\int \frac{d\mu}{\mu} = \int -1\,dx,$$

so that

$$\ln(|\mu|) = -x.$$

We have taken $C = 0$, because we are not looking for the most general integrating factor. Any solution of equation (3) will do. Finally, exponentiating both sides, we have

$$\mu(x) = e^{-x}$$

as the integrating factor.

The converted exact equation is then

$$(1 - e^{-x}\sin(y)) + e^{-x}\cos(y) = 0.$$

This can be checked to be exact — $\partial M/\partial y = \partial N/\partial x = -e^{-x}\cos(y)$. Solutions are given implicitly by

$$x + e^{-x}\sin(y) = C.$$

iv) For the equation $xy^2 + x\frac{dy}{dx} = 0$, we have

$$\begin{aligned}\frac{\partial M}{\partial y} &= 2xy,\\ \frac{\partial N}{\partial x} &= 1,\\ N &= x, \quad \text{and}\\ \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} &= \frac{2xy-1}{x},\end{aligned}$$

which is not a function of x only.

2. i) Check that

$$W(e^t, te^t, t^2e^t)(t) = \det\begin{pmatrix} e^t & te^t & t^2e^t \\ (e^t)' & (te^t)' & (t^2e^t)' \\ (e^t)'' & (te^t)'' & (t^2e^t)'' \end{pmatrix} = 2e^{3t}, \quad t \in \mathbb{R},$$

and conclude that $\{e^t, te^t, t^2e^t\}$ is a linearly independent subset of $C^\infty(\mathbb{R}, \mathbb{R})$.

ii) Show that if

$$\alpha_1 e^t + \alpha_2 te^t + \alpha_3 t^2 e^t = 0$$

for all $t \in \mathbb{R}$, then $\alpha_1 = \alpha_2 = \alpha_3 = 0$ (give a direct argument avoiding the use of Wronskians). This gives another argument that $\{e^t, te^t, t^2e^t\}$ is a linearly independent subset of $C^\infty(\mathbb{R}, \mathbb{R})$.

Solution. i) First, we compute the derivatives involved.

$$\begin{aligned} (e^t)' &= e^t & \quad\text{and}\quad && (e^t)'' &= e^t, \\ (te^t)' &= e^t + te^t = (1+t)e^t & \quad\text{and}\quad && (e^t)'' &= e^t + (1+t)e^t = (2+t)e^t, \\ (t^2e^t)' &= 2te^t + t^2e^t = (2t+t^2)e^t & \quad\text{and}\quad && (t^2e^t)'' &= (2+2t)e^t + (2t+t^2)e^t = (2+4t+t^2)e^t. \end{aligned}$$

Now, the Wronskian is

$$\det\begin{pmatrix} e^t & te^t & t^2e^t \\ e^t & (1+t)e^t & (2t+t^2)e^t \\ e^t & (2+t)e^t & (2+4t+t^2)e^t \end{pmatrix} = (e^t)^3 \det\begin{pmatrix} 1 & t & t^2 \\ 1 & 1+t & 2t+t^2 \\ 1 & 2+t & 2+4t+t^2 \end{pmatrix}.$$

We can simplify the computation by subtracting the first row from both the second and the third row (this does not change the determinant), obtaining

$$e^{3t} \det\begin{pmatrix} 1 & t & t^2 \\ 0 & 1 & 2t \\ 0 & 2 & 2+4t \end{pmatrix}.$$

Expanding the result along the first column, the Wronskian is equal to

$$e^{3t} \det\begin{pmatrix} 1 & 2t \\ 2 & 2+4t \end{pmatrix} = e^{3t}(2+4t-4t) = 2e^{3t}.$$

Because

$$W(e^t, te^t, t^2e^t)(t) \neq 0 \quad \text{for all } t,$$

we conclude that $\{e^t, te^t, t^2e^t\}$ is a linearly independent set (it is enough that the Wronskian does not vanish for a *fixed* t, but here it does not vanish for all t).

ii) Suppose that

$$\alpha_1 e^t + \alpha_2 te^t + \alpha_3 t^2 e^t = 0. \tag{4}$$

Evaluating at $t = 0$, we have

$$\alpha_1 \cdot 1 + \alpha_2 \cdot 0 + \alpha_3 \cdot 0 = 0,$$

so $\alpha_1 = 0$.

Differentiating eq. 4 with respect to t, we have

$$0 + \alpha_2 (1+t)e^t + \alpha_3 (2t+t^2)e^t = 0.$$

Evaluating the result at $t = 0$, we have

$$\alpha_2 \cdot 1 + \alpha_3 \cdot 0 = 0,$$

so $\alpha_2 = 0$.

Differentiating once more, we have

$$0 + \alpha_3(2 + 4t + t^2)e^t = 0.$$

Evaluating at $t = 0$, we conclude that $\alpha_3 = 0$.

Alternative solution. Instead, we could have evaluated eq. 4 at two more points, say $t = 1$ and $t = 2$, obtaining the system of equations

$$\begin{aligned} \alpha_2\, e + \alpha_3\, e &= 0 \\ \alpha_2\, 2e^2 + \alpha_3\, 4e^2 &= 0, \end{aligned}$$

or, equivalently,

$$\begin{aligned} \alpha_2 + \alpha_3 &= 0 \\ 2\alpha_2 + 4\alpha_3 &= 0. \end{aligned}$$

This is easily seen to have solution $\alpha_2 = \alpha_3 = 0$. (For instance, we could subtract two times the first equation from the second, obtaining $2\alpha_3 = 0$, hence $\alpha_2 = 0$ from the first equation.)

3. Let V and W be finite-dimensional vector spaces. Show that if there exists an isomorphism $L\colon V \to W$, then $\dim V = \dim W$.

(This fact was needed in the proof that the dimension of the space of solutions of a homogeneous linear equation is equal to its order. It is a particular case of the principle that isomorphic vector spaces have identical linear-algebraic properties.)

Suggestion: Let $\dim V = r$. Choose a basis $\{e_1, \dots, e_r\}$ of V. Show that $\{L(e_1), \dots, L(e_r)\}$ is a basis of W.

Solution. Following the suggestion, let $\dim V = r$, and choose a basis $\{e_1, \dots, e_r\}$ of V.

Claim. $\{L(e_1), \dots, L(e_r)\}$ is a basis of W.

Proof of Claim. By definition of basis, the claim will be shown if we check that $\{L(e_1), \dots, L(e_r)\}$ is a linearly independent set, and that it spans all of W.

Suppose that

$$\alpha_1 L(e_1) + \cdots + \alpha_r L(e_r) = 0$$

for some $\alpha_1, \dots, \alpha_r$. By linearity of L, the left side is equal to $L(\alpha_1 e_1 + \cdots + \alpha_r e_r)$. Because L is an isomorphism, it is injective, and so the only vector sent to 0 is 0. Thus,

$$\alpha_1 e_1 + \cdots + \alpha_r e_r = 0.$$

Because $\{e_1, \dots, e_r\}$ is a basis of V, it is a linearly independent set. We conclude that $\alpha_1 = \alpha_2 = \cdots = \alpha_r = 0$. Therefore, $\{L(e_1), \dots, L(e_r)\}$ is a linearly independent set.

To show that $\{L(e_1), \ldots, L(e_r)\}$ spans W, choose any element w of W. Because L is an isomorphism, it is surjective, so there exists an element v of V with $L(v) = w$. Because $\{e_1, \ldots, e_r\}$ is a basis of V, there exist $\alpha_1, \ldots, \alpha_r$ such that

$$v = \alpha_1 e_1 + \cdots + \alpha_r e_r.$$

Applying L, we have

$$w = L(v) = L(\alpha_1 e_1 + \cdots + \alpha_r e_r) = \alpha_1 L(e_1) + \cdots + \alpha_r L(e_r).$$

Therefore, w is in the span of $\{L(v_1), \ldots, L(v_r)\}$. Since w was arbitrary, $\{L(v_1), \ldots, L(v_r)\}$ is a spanning set of W.

We conclude that $\{L(v_1), \ldots, L(v_r)\}$ is a basis of W. $\square$

Because V and W have bases with an equal number of elements, they have equal dimension.

Alternative solution. The rank-nullity theorem says that for any linear map $L\colon V \to W$ between finite-dimensional vector spaces[2], we have

$$\dim \operatorname{im} L + \dim \ker L = \dim V.$$

(The number $\dim \operatorname{im} L$ is called the rank of L and the number $\dim \ker L$ is called the nullity of L, which explains the name of the theorem.)

When L is an isomorphism,

$$\begin{aligned} \dim \operatorname{im} L &= \dim W, \quad \text{as } L \text{ is surjective, and} \\ \dim \ker L &= 0, \quad \text{as } L \text{ is injective.} \end{aligned}$$

Therefore, $\dim W + 0 = \dim V$, which is what we wanted to show.

Optional Problem. Let I be the open interval $(-1, 1)$. Can you find a function in $C^0(I, \mathbb{R})$ but not $C^1(I, \mathbb{R})$? In $C^1(I, \mathbb{R})$ but not $C^2(I, \mathbb{R})$? In $C^r(I, \mathbb{R})$ but not $C^{r+1}(I, \mathbb{R})$?

Solution. A function is in $C^0(I, \mathbb{R})$ but not $C^1(I, \mathbb{R})$ if it continuous over I, but not differentiable (or does not have a continuous derivative) at some points of I.

The first example that comes to mind is the absolute value function,

$$\phi(t) = |t| = \begin{cases} t, & 0 \le t < 1 \\ -t, & -1 < t < 0 \end{cases}$$

It is continuous, not differentiable at $t = 0$.

Now, by the fundamental theorem of calculus, if $f(t)$ is continuous over an open interval I, then the function $F(t)$ defined by

$$F(t) = \int_{t_0}^{t} f(x)\, dx, \qquad t_0 \in I$$

[2] Actually, W may be infinite-dimensional.

is *differentiable* on I, with derivative $F'(t) = f(t)$. Since f is continuous on I, we see that F has a continuous derivative on I.

Iterating this construction r times with f being the absolute value function, we obtain a function in $C^r(I, \mathbb{R})$, but not in $C^{r+1}(I, \mathbb{R})$:

$$\phi_r(t) = \underbrace{\int_{-1}^{t} \cdots \int_{-1}^{t}}_{r \text{ integrals}} |x_1| \, dx_1 \, dx_2 \cdots dx_r.$$

Explicitly, the first few iterations produce

$$\begin{aligned}
\phi_0(t) &= |t|, \\
\phi_1(t) &= \frac{1 + t\,|t|}{2}, \\
\phi_2(t) &= \frac{2 + 3t + t^2\,|t|}{6}.
\end{aligned}$$

Now, we notice that some of the terms produced by the procedure above are not essential for making sure a function only has derivatives up to a certain order. Discarding the inessential terms then leads us to the following family of examples:

$$\phi_r(t) = \frac{t^r\,|t|}{(r+1)!},$$

where $r! = 1 \cdot 2 \cdot 3 \cdots (r-1) \cdot r$ and, by convention, $0! = 1$.

Claim. We have

$$\frac{d}{dt}\phi_r(t) = \phi_{r-1}(t) \quad \textit{for all } r \geq 1.$$

Proof of Claim. For $t > 0$, we have

$$\phi_r(t) = \frac{t^r \cdot t}{(r+1)!}, \quad \text{so} \quad \frac{d}{dt}\phi_r(t) = (r+1)\frac{t^r}{(r+1)!} = \frac{t^r}{r!} = \frac{t^{r-1} \cdot |t|}{r!} = \phi_{r-1}(t).$$

For $t < 0$, we have

$$\phi_r(t) = \frac{t^r \cdot -t}{(r+1)!}, \quad \text{so} \quad \frac{d}{dt}\phi_r(t) = -(r+1)\frac{t^r}{(r+1)!} = -\frac{t^r}{r!} = \frac{t^{r-1} \cdot |t|}{r!} = \phi_{r-1}(t).$$

For $t = 0$, we proceed from the definition of derivative:

$$\lim_{h \to 0} \frac{\phi_r(0+h) - \phi_r(0)}{h} = \frac{\frac{h^r|h|}{(r+1)!}}{h} = \lim_{h \to 0} \frac{h^{r-1}\,|h|}{(r+1)!} = 0 = \phi_{r-1}(0).$$

□

Since $\phi_0(t) = |t|$ is not differentiable, we see that ϕ_r is r-times differentiable, but not $r+1$-times differentiable, as desired.