MTHE 237 — PROBLEM SET 02 SOLUTIONS

1. Solve the following differential equations.
(Hand-in only the starred problems, but please attempt them all. Solutions can be left in implicit form. The integral $\int \frac{dx}{1+x^2} = \arctan(x)$ may be useful. There are suggestions at the bottom of p. 2, but try solving without looking at the suggestions first.)

i) $(3x^2y + 8xy^2) + (x^3 + 8x^2y + 12y^2)\frac{dy}{dx} = 0, \quad y(1) = 0.$

ii)* $\frac{dy}{dx} = x^2 + 2xy + y^2, \quad y(0) = 0.$

iii)* $(x - y)\frac{dy}{dx} = (x + y), \quad y(1) = 0.$

iv) $(x - y - 1)\frac{dy}{dx} = (x + y + 1), \quad y(1) = -1.$

v) $\frac{dy}{dx} = e^{x+y}, \quad y(0) = 0.$

vi) $\frac{x}{\sqrt{x^2+y^2}} + \frac{y}{\sqrt{x^2+y^2}}\frac{dy}{dx} = 0, \quad y(1) = 0.$

vii)* $(x - 2xy + e^y) + (y - x^2 + xe^y)\frac{dy}{dx} = 0, \quad y(0) = 1.$

viii) $x\frac{dy}{dx} = xe^{y/x} + y, \quad y(1) = 0.$

Solution. i) Suspecting the equation is exact, we check that $\partial M/\partial y = \partial N/\partial x$:

$$\partial M/\partial y = 3x^2 + 16xy,$$
$$\partial N/\partial x = 3x^2 + 16xy + 0.$$

Since $M, N, \partial M/\partial y$ and $\partial N/\partial x$ are all continuous on $\mathbb{R}^2$, and we have the equality $\partial M/\partial y = \partial N/\partial x$, the equation is exact.

Integrating $M(x,y) = 3x^2y + 8xy^2$ with respect to x, we get

$$x^3y + 4x^2y^2 + h(y).$$

Taking the partial with respect to y, we get

$$x^3 + 8x^2y + h'(y).$$

Comparing with $N(x,y)$, we see that

$$h'(y) = 12y^2,$$

so that

$$h(y) = 4y^3 + C.$$

Implicit solutions to the equation are therefore level curves

$$x^3y + 4x^2y^2 + 4y^3 = C.$$

To find C, we apply the initial condition—

$$1^3 \cdot 0 + 4 \cdot 1^2 \cdot 0^2 + 4 \cdot 0^3 = C,$$

so that $C = 0$ and the solution is given implicitly by

$$x^3y + 4x^2y^2 + 4y^3 = 0.$$

ii) As a preliminary step, we factor the right side: $x^2 + 2xy + y^2 = (x+y)^2$. This suggests the change of variable $v(x) = x + y(x)$. Solving for y in terms of x and v, we have

$$y = v - x, \quad \frac{dy}{dx} = \frac{dv}{dx} - 1,$$

The equation rewritten in terms of v and x is

$$\left(\frac{dv}{dx} - 1\right) = v^2,$$
$$\frac{dv}{dx} = v^2 + 1,$$

which is now separable.

$$\frac{1}{v^2+1}\frac{dv}{dx} = 1.$$

Integrating both sides, solutions are given implicitly by

$$\int \frac{dv}{v^2+1} = x + C.$$

The integral on the left is $\arctan(v)$. Applying tan to both sides, we have

$$v = \tan(x + C).$$

Rewriting in terms of y, we have

$$x + y = \tan(x + C),$$

so that

$$y(x) = \tan(x + C) - x.$$

Applying the initial condition, we find that $C = 0$, so that the solution is

$$y(x) = \tan(x) - x, \quad -\pi/2 < x < \pi/2.$$

The restrictions on the domain arise from the requirement that solutions be continuous functions, and the non-removable discontinuities of tan at $x = \pi/2 + k\pi$, k an integer.

iii) This equation is homogeneous. We use the standard change of variable $v(x) = y(x)/x$. The induced initial condition is $v(1) = 0/1 = 0$. We need to restrict to $x > 0$ for this change of variable. We have, as usual,

$$y = vx, \quad \frac{dy}{dx} = v + x\frac{dv}{dx}.$$

Rewriting the equation in terms of x and v,

$$\begin{aligned}
(x - xv)\left(v + x\frac{dv}{dx}\right) &= (x + xv), \\
x(1 - v)\left(v + x\frac{dv}{dx}\right) &= x(1 + v), \\
v + x\frac{dv}{dx} &= \frac{1+v}{1-v}, \\
x\frac{dv}{dx} &= \frac{1+v}{1-v} - v, \\
x\frac{dv}{dx} &= \frac{1 + v - v + v^2}{1-v} = \frac{1+v^2}{1-v},
\end{aligned}$$

so that we have the separated equation

$$\frac{1-v}{1+v^2}\,\frac{dv}{dx} = \frac{1}{x}.$$

Integrating the left side, we have

$$\int \frac{1-v}{1+v^2}\,dv = \int \frac{dv}{1+v^2} - \int \frac{v}{1+v^2}\,dv = \arctan(v) - \frac{1}{2}\ln(1+v^2) = \arctan(v) - \ln(\sqrt{1+v^2}),$$

so that the implicit solutions are given by

$$\arctan(v) - \ln(\sqrt{1+v^2}) = \ln(|x|) + C.$$

Since $v(1) = 0$ is the initial condition, we see that $C = 0$. Rewriting in terms of x and y, we obtain the implicit solution

$$\arctan(y/x) - \ln(\sqrt{1 + (y/x)^2}) = \ln(x), \quad x > 0.$$

iv) This equation looks very similar to the previous one. We make the change-of-variable $v = y + 1,\ v(1) = y(1) + 1 = -1 + 1 = 0$ so that

$$y = v - 1, \quad \frac{dy}{dx} = \frac{dv}{dx}.$$

and the equation becomes

$$(x - v)\,\frac{dv}{dx} = x + v, \quad v(1) = 0.$$

This equation was solved in problem iii), the implicit solution is

$$\arctan(v/x) - \ln(\sqrt{1+(v/x)^2}) = \ln(x), \quad x > 0.$$

Rewriting in terms of x and y, we have

$$\arctan((y+1)/x) - \ln(\sqrt{1+((y+1)/x)^2}) = \ln(x), \quad x > 0$$

as the implicit solution.

v) This is separable, since $e^{x+y} = e^x e^y$! We have

$$e^{-y}\frac{dy}{dx} = e^x.$$

Integrating both sides,

$$-e^{-y} = e^x + C.$$

The initial condition is $y(0) = 0$, so $-1 = 1 + C$, which implies that $C = -2$. The implicit solution is

$$e^x + e^{-y} = 2.$$

vi) This equation is exact. We have:

$$\begin{aligned} \partial M/\partial y &= (-1/2)\frac{x}{(x^2+y^2)^{3/2}}\,2y = -\frac{xy}{(x^2+y^2)^{3/2}}, \\ \partial N/\partial x &= (-1/2)\frac{y}{(x^2+y^2)^{3/2}}\,2x = -\frac{xy}{(x^2+y^2)^{3/2}}. \end{aligned}$$

Integrating $M(x,y) = \frac{x}{\sqrt{x^2+y^2}}$ with respect to x, we have

$$\sqrt{x^2+y^2} + h(y).$$

Taking the partial with respect to y, we have

$$\frac{y}{\sqrt{x^2+y^2}} + h'(y).$$

Comparing with $N(x,y)$, we see that $h'(y) = 0$, so that $h(y) = C$. The implicit solutions are

$$\sqrt{x^2+y^2} = C.$$

From the initial condition $y(1) = 0$, we see that $C = 1$, so that the implicit solution is

$$\sqrt{x^2+y^2} = 1.$$

vii) Testing for exactness, we compute

$$\begin{aligned} \partial M/\partial y &= -2x + e^y, \\ \partial N/\partial x &= -2x + e^y. \end{aligned}$$

The partials agree and $M, N, \partial M/\partial y$ and $\partial N/\partial x$ are continuous on $\mathbb{R}^2$, so we conclude that the equation is exact.

Integrating with $M(x,y) = x - 2xy + e^y$ with respect to x, we have

$$\frac{1}{2}x^2 - x^2y + xe^y + h(y).$$

Taking the partial with respect to y, we get

$$-x^2 + xe^y + h'(y).$$

Comparing with $N(x,y) = y - x^2 + xe^y$, we see

$$h'(y) = y, \quad \text{so} \quad h(y) = \frac{1}{2}y^2 + C.$$

Implicit solutions are

$$\frac{1}{2}(x^2 + y^2) - x^2y + xe^y = C.$$

The initial condition implies that $C = 1/2$, so the implicit solution is

$$\frac{1}{2}(x^2 + y^2) - x^2y + xe^y = \frac{1}{2}.$$

viii) The equation is homogeneous (the term $e^{y/x}$ is fairly sneaky). Applying the usual change of variable $v = y/x$, $v(1) = 0/1 = 0$, and

$$y = vx, \quad \frac{dy}{dx} = v + x\frac{dv}{dx},$$

the equation becomes

$$\begin{aligned} x\left(v + x\frac{dv}{dx}\right) &= xe^{vx/x} + vx, \\ v + x\frac{dv}{dx} &= e^v + v, \\ x\frac{dv}{dx} &= e^v. \end{aligned}$$

Separating variables,

$$e^{-v}\frac{dv}{dx} = \frac{1}{x}.$$

Integrating,

$$-e^{-v} = \ln(|x|) + C.$$

Using the initial condition $v(1) = 0$, we see that $C = -1$. The implicit solution is

$$\ln(x) + e^{-v} = 1.$$

Rewriting in terms of x and y, we get

$$\ln(x) + e^{-y/x} = 1, \quad x > 0.$$

2. For each of the following differential equations, is existence of a solution in some nonempty open interval about x_0 implied by the Existence and Uniqueness Theorem for First-Order Ordinary Differential Equations? If so, is uniqueness?

i) $\dfrac{dy}{dx} = x^2 + 2xy + y^2, \quad y(0) = 0.$

ii) $\dfrac{dy}{dx} = \dfrac{1}{x^2 + y^2 + 1}, \quad y(0) = 1.$

iii) $\dfrac{dy}{dx} = y^{1/5}, \quad y(1) = 1.$

iv) $\dfrac{dy}{dx} = y^{1/5}, \quad y(2) = 0.$

Solution. i) The functions

$$F(x,y) = x^2 + 2xy + y^2,$$
$$\frac{\partial F}{\partial y}(x,y) = 2x + 2y$$

are polynomials in x and y, and are therefore continuous on all of $\mathbb{R}^2$. By the Existence-Uniqueness theorem, a unique solution through $(0,0)$ exists in a (nonempty) open interval about $x = 0$.

ii) The functions

$$F(x,y) = \frac{1}{x^2 + y^2 + 1},$$
$$\frac{\partial F}{\partial y}(x,y) = -\frac{2y}{(x^2 + y^2 + 1)^2}$$

are continuous on all of $\mathbb{R}^2$. By the Existence-Uniqueness theorem, a unique solution through $(0,1)$ exists in a (nonempty) open interval about $x = 0$.

iii) The functions are

$$F(x,y) = y^{1/5},$$
$$\frac{\partial F}{\partial y}(x,y) = \frac{1}{5}y^{-4/5}.$$

$F(x,y)$ is continuous on all of $\mathbb{R}^2$; $\partial F/\partial y$ is not defined for $y = 0$ and is continuous elsewhere.

We can choose a small rectangle containing $(1,1)$ that does not intersect the line $y = 0$. Therefore, both parts of the existence and uniqueness theorem apply. There exists a unique solution through $(1,1)$ in a (nonempty) open interval about $x = 1$.

iv) The same functions as part iii). Existence part of the theorem applies, but the solution may not be unique, because any rectangle containing $(2,0)$ will contain points of discontinuity of $y^{-4/5}$ (which are, as explained in the previous part, along the line $y = 0$).

3. In this problem, we look at water flowing out of a container through an opening at its bottom.

Let $h(t)$ denote the height of the water level above the bottom of the container at time t, let $A(t)$ denote the area of the top surface of the water at time t, and let a denote the cross-sectional area of the opening at the bottom. It follows from conservation of energy (Torricelli's principle) that

$$A(t)\,\frac{dh}{dt}(t) = -a\sqrt{2g\,h(t)}, \tag{1}$$

where g is the gravitational constant.

Consider three possible containers—

i) Solve eq. (1) for a cylindrical container of height h_0 with cross-sectional area A (so that $A(t) = A$ is a constant function) that is standing upright (that is, on the circular face that has the opening). The initial condition is that the container is full at $t = 0$, so that $h(0) = h_0$.

ii) Now consider a circular cone of height h_0 and with diameter $2h_0$ across the top, standing upright on its vertex. Using similar triangles, argue that $A(t) = \pi h(t)^2$. Then, solve eq. (1) with initial condition $h(0) = h_0$.

iii) A paraboloid of revolution is the surface obtained by rotating the graph of the function $y(x) = x^2$ about the y-axis. For a container of height h_0 in the shape of a paraboloid of revolution, find a relationship between $A(t)$ and $h(t)$, and solve eq. (1). The initial condition is once again $h(0) = h_0$.

iv) In terms of h_0, how much time does it take for each of the three containers to completely empty out?

v) Supposing $a\sqrt{2g} = 1$ and $h_0 = 1$, sketch a plot of $h(t)$ versus t for each of the three containers (you may need to use a plotting program for this).

Solution. i) The equation is separable. Separating variables, we get

$$A\,h^{-1/2}\,\frac{dh}{dt} = -a\sqrt{2g}$$

Integrating both sides, we find

$$2Ah^{1/2} = -a\sqrt{2g}\,t + C.$$

Solving for h as a function of t,

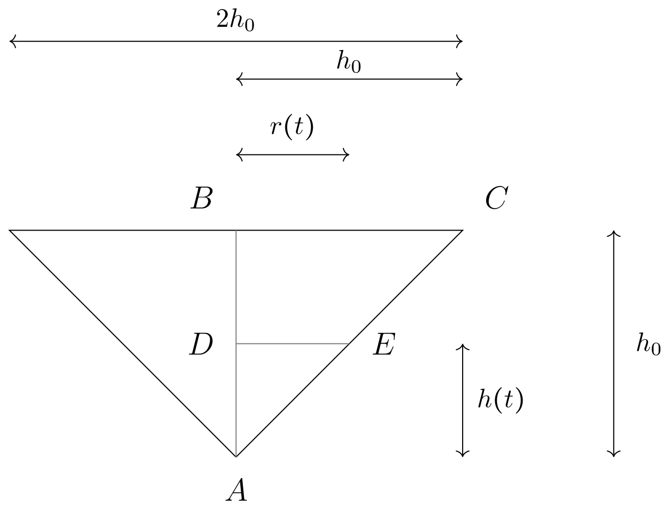
$$h(t) = \left(C - \frac{a}{A}\sqrt{\frac{g}{2}}\,t\right)^2.$$

The initial condition is $h(0) = h_0$. This implies that $C = \sqrt{h_0}$, and the function we are looking for is

$$h(t) = \left(\sqrt{h_0} - \frac{a}{A}\sqrt{\frac{g}{2}}\,t\right)^2, \quad 0 \le t < \frac{A}{a}\sqrt{\frac{2}{g}}\sqrt{h_0}.$$

The restrictions on the domain of t come from physical considerations: the water level should not overflow the tank, and cannot be negative (the strict inequality in the upper bound comes from the process of separating variables, since we divided by $h^{1/2}$).

ii) Here is a sketch of a vertical cross-section of the container:



The trianges ABC and ADE are similar. Denoting the radius of the circle of the cross-section at height $h(t)$, we see that

$$\frac{r(t)}{h(t)} = \frac{h_0}{h_0} = 1, \qquad \text{so } r(t) = h(t).$$

Applying the formula for the area of a circle in terms of its radius, we find that $A(t) = \pi h(t)^2$. Eq. 1 becomes

$$\pi h^{3/2} \frac{dh}{dt} = -a\sqrt{2g}.$$

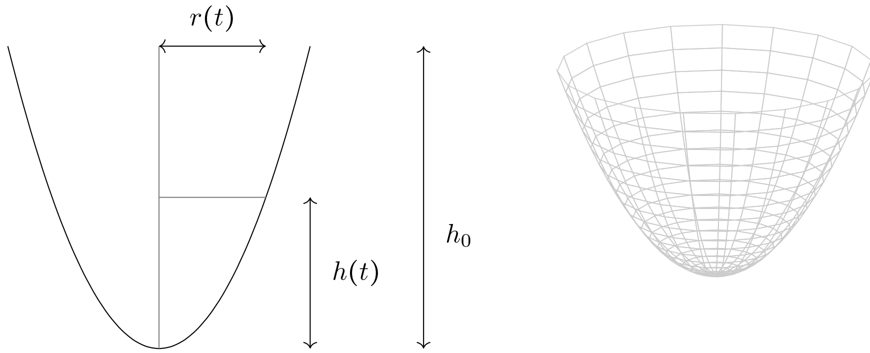
Integrating both sides,

$$\frac{2\pi}{5} h^{5/2} = C - a\sqrt{2g}\, t.$$

Solving for h as a function of t and applying the initial condition,

$$h(t) = \left(h_0^{5/2} - \frac{5a}{2\pi}\sqrt{2g}\, t \right)^{2/5}, \quad 0 \le t < \frac{2\pi h_0^{5/2}}{5a\sqrt{2g}}.$$

iii) Again, we sketch a vertical cross-section of the container:



At height $h(t)$, the radius $r(t)$ of the horizontal cross-section of the container is equal to $\sqrt{h(t)}$. To see this, notice that the point $(r(t), h(t))$ lies on the graph of the parabola $y = x^2$. So, $h(t) = r(t)^2$ and $r(t) = \sqrt{h(t)}$. Therefore, $A(t) = \pi h(t)$. Eq. 1 becomes

$$\pi h^{1/2} \frac{dh}{dt} = -a\sqrt{2g}.$$

Integrating both sides,

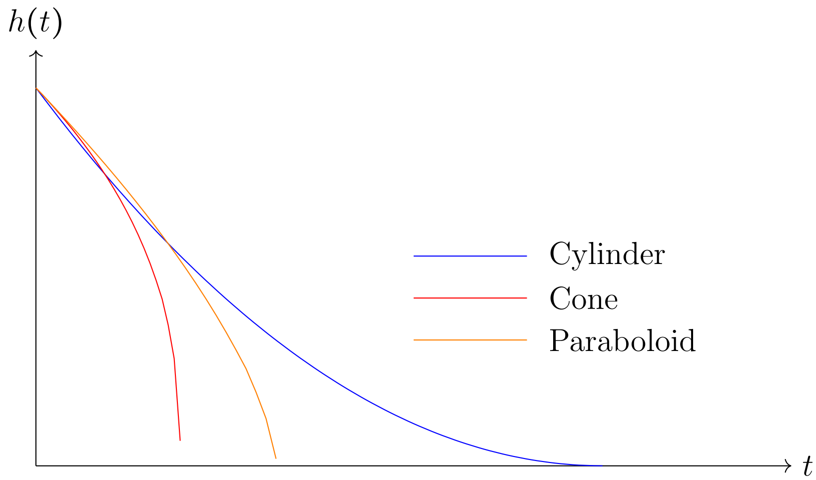
$$\frac{2\pi}{3} h^{3/2} = C - a\sqrt{2g}\, t.$$

Solving for h as a function of t and applying the initial condition,

$$h(t) = \left(h_0^{3/2} - \frac{3a}{2\pi}\sqrt{2g}\, t\right)^{2/3}, \quad 0 \le t < \frac{2\pi h_0^{3/2}}{3a\sqrt{2g}}.$$

iv) $t = \frac{A}{a}\sqrt{\frac{2}{g}}\sqrt{h_0}$, $\quad t = \frac{2\pi h_0^{5/2}}{5a\sqrt{2g}}$ and $t = \frac{2\pi h_0^{3/2}}{3a\sqrt{2g}}$, respectively.

v) We take $A = 1/2$ for the plot of the cylindrical container.



Optional Problem. The purpose of this question is to prove that the differential equation $dy/dx = x + y$ is not separable. (As a reminder, a differential equation is separable if it can be written in the form $n(y)\,dy/dx = m(x)$, or, equivalently, in the form $dy/dx = m(x)/n(y)$.)

i) Let $F(x, y)$ be a real-valued function of two variables. Show that if $F(x, y) = m(x)/n(y)$ for some functions m, n of a single variable, then for any real numbers x_0, x_1, y_0, y_1 such that the four points (x_0, y_0), (x_0, y_1), (x_1, y_1), (x_1, y_1) are in the domain of F,

$$F(x_0, y_0)F(x_1, y_1) - F(x_0, y_1)F(x_1, y_0) = 0.$$

ii) Conclude that $F(x, y) = x + y$ cannot be written in the form $F(x, y) = m(x)/n(y)$.

Solution. i) This is a simple matter of plugging in $F(x, y) = m(x)/n(y)$:

$$F(x_0, y_0)F(x_1, y_1) - F(x_0, y_1)F(x_1, y_0) = \frac{m(x_0)}{n(y_0)}\frac{m(x_1)}{n(y_1)} - \frac{m(x_0)}{n(y_1)}\frac{m(x_1)}{n(y_0)} = 0.$$

ii) Trying the criterion of part i) with the function $F(x, y) = x + y$, we find

$$\begin{aligned} F(x_0, y_0)F(x_1, y_1) - F(x_0, y_1)F(x_1, y_0) &= (x_0 + y_0)(x_1 + y_1) - (x_0 + y_1)(x_1 + y_0) \\ &= x_0x_1 + x_0y_1 + y_0x_1 + y_0y_1 - (x_0x_1 + x_0y_0 + y_1x_1 + y_1y_0) \\ &= x_0y_1 + y_0x_1 - x_0y_0 - x_1y_1. \end{aligned}$$

Pick $x_0 = 0$, $x_1 = 1$, $y_0 = 1$, $y_1 = 0$. The expression becomes

$$0 \cdot 1 + 1 \cdot 1 - 0 \cdot 1 - 1 \cdot 0 = 1 \neq 0.$$

Since the criterion of part i) is not met, we can conclude that $F(x, y) = x + y$ cannot be written in the form $F(x, y) = m(x)/n(y)$.

Suggestions for ***1.*** i) Exact; ii) Factor and try substitution $v = x + y$; iii) Homogeneous; iv) Substitute to reduce to iii; v) Separable; vi); Exact; vii) Exact; viii) Homogeneous.