MTHE 237 — PROBLEM SET 01 SOLUTIONS

Remark. In the following answers, we explicitly keep track of the restrictions on the domain of a solution of a differential equation that arise in the process of separating variables. In future assignments, we will allow ourselves not to keep track of these restrictions. (Students will not be required keep track of the restrictions, for any homework including the first).

1. For each of the following, find a function $y(x)$ that satisfies the differential equation and initial condition. Take care to provide the domain of definition of the solution.

i) $\dfrac{dy}{dx} = 5y^2, \quad y(0) = 1;$

ii) $x\dfrac{dy}{dx} = 2(y - 9), \quad y(1) = 10;$

iii) $(x^2 + 1)\dfrac{dy}{dx} = xy, \quad y(0) = 1;$

iv) $\dfrac{dy}{dx} = xy + 2x + y + 2, \quad y(0) = -1;$

v) $2xy\dfrac{dy}{dx} = (x^2 + y^2), \quad y(1) = \sqrt{3}.$

Solution. i) Separating variables, we have

$$\frac{1}{y^2}\frac{dy}{dx} = 5, \quad y \neq 0.$$

Because we divided by y^2, we have lost the constant solution $y(x) = 0$. This solution does not satisfy the initial condition $y(0) = 1$, however, so we may disregard this solution and proceed with the algorithm.

By our algorithm, solutions are given implicitly by

$$\int \frac{dy}{y^2} = \int 5\,dx.$$

Doing the integrals, we get

$$-\frac{1}{y} = 5x + C,$$

so that, solving for y,

$$y(x) = -\frac{1}{5x + C}.$$

Using the initial condition, we find the particular solution:

$$1 = y(0) = -\frac{1}{5 \cdot 0 + C} = -\frac{1}{C}, \qquad \text{so } C = -1 \quad \text{and} \quad y(x) = -\frac{1}{5x - 1} = \frac{1}{1 - 5x}.$$

Since $\frac{1}{1-5x}$ has a non-removable discontinuity at $x = \frac{1}{5}$ (and is defined and differentiable elsewhere), the possible domains of the solution are the intervals $x < \frac{1}{5}$ and $x > \frac{1}{5}$. The interval that contains $x = 0$ is $x < \frac{1}{5}$, so this is the domain of the solution.

Finally, we see that $y(x) > 0$ for all $x < \frac{1}{5}$, so the requirement that $y \neq 0$ puts no additional restrictions on the domain.

$$y(x) = \frac{1}{1-5x}, \quad x < \frac{1}{5}.$$

ii) Separating variables, we have

$$\frac{1}{y-9}\frac{dy}{dx} = \frac{2}{x}, \quad y \neq 9,\ x \neq 0.$$

Because we have divided by $y - 9$, we have lost the constant solution $y(x) = 9$. This solution does not satisfy the initial condition $y(1) = 10$, however, so we may disregard this solution and proceed.

By the algorithm, solutions are given implicitly by

$$\int \frac{dy}{y-9} = \int \frac{2}{x}\, dx.$$

Integrating,

$$\ln(|y-9|) = 2\ln(|x|) + C = \ln\left(\left|x^2\right|\right) + C.$$

Taking exponentials of both sides,

$$y - 9 = Cx^2 \quad \text{so} \quad y(x) = 9 + Cx^2.$$

Using the initial condition to pick out the particular solution,

$$10 = y(1) = 9 + C \cdot 1^2, \quad \text{so } C = 1.$$

The function $y(x) = 9 + x^2$ crosses the restriction $y \neq 9$, $x \neq 0$ at $(0, 9)$, hence we get the additional restriction $x > 0$ on the domain.

$$y(x) = 9 + x^2, \quad x > 0.$$

Remark. For this question, the hypotheses of the existence and uniqueness theorem do not hold over the y-axis, and in fact there is a two-parameter family of solutions passing through $(0, 9)$.

iii) Separating variables, we have

$$\frac{1}{y}\frac{dy}{dx} = \frac{x}{x^2+1}, \quad y \neq 0.$$

Because we have divided by y, we have lost the constant solution $y(x) = 0$. This solution does not satisfy the initial condition $y(0) = 1$, however. We disregard this solution.

By the algorithm, solutions are given implicitly by

$$\int \frac{dy}{y} = \int \frac{x}{x^2+1}\,dx$$

Integrating,

$$\ln(|x|) = \frac{1}{2}\ln(\left|x^2+1\right|) + C = \ln(\sqrt{x^2+1}) + C.$$

(Note that $x^2+1>0$ for all x, which justifies omitting the absolute value.) Taking exponentials of both sides,

$$y = C\sqrt{x^2+1}.$$

To satisfy the initial condition,

$$1 = y(0) = C\sqrt{0^2+1} = C, \quad \text{so } C = 1.$$

The function $y(x) = \sqrt{x^2+1}$ is defined and differentiable for all x. Moreover, we have $y(x) > 0$ for all x, so the condition $y \neq 0$ does not impose any additional restrictions on the domain of the solution. The solution is

$$y(x) = \sqrt{x^2+1}, \quad x \in \mathbb{R}.$$

iv) As a preliminary step, we notice that the right-hand side factors as

$$xy + 2x + y + 2 = x(y+2) + y + 2 = (x+1)(y+2).$$

Therefore, the equation is

$$\frac{dy}{dx} = (x+1)(y+2), \quad y(0) = -1.$$

Separating variables, we have

$$\frac{1}{y+2}\frac{dy}{dx} = x+1, \quad y \neq -2.$$

Dividing by $y+2$ lost the constant solution $y(x) = -2$. However this does not satisfy the initial condition, so we disregard it.

By the algorithm, solutions are given implicitly by

$$\int \frac{dy}{y+2} = \int x+1\,dx.$$

Integrating,

$$\ln(|y+2|) = \frac{x^2}{2} + x + C.$$

Taking exponentials of both sides,

$$y+2 = C\,e^{\frac{x^2}{2}+x}, \text{ so that } y = C\,e^{\frac{x^2}{2}+x} - 2.$$

To satisfy the initial condition, we need

$$-1 = Ce^0 - 2 = C - 2, \quad \text{so } C = 1.$$

The function $y(x) = e^{x^2/2+x} - 2$ is defined and differentiable for all x. Moreover, $y(x) > -2$, so the condition $y \neq -2$ imposes no additional restrictions on the domain of the solution. So the solution is

$$y(x) = e^{\frac{x^2}{2}+x} - 2, \quad x \in \mathbb{R}.$$

v) This differential equation is not separable, but it is homogeneous of degree 2. We therefore make the substitution

$$y = xv, \quad \frac{dy}{dx} = v + x\frac{dv}{dx}.$$

The substitution places the restriction $x \neq 0$ on the domain of a solution. The initial condition $y(1) = \sqrt{3}$ determines an initial condition for v: $\sqrt{3} = y(1) = 1 \cdot v(1)$ and so $v(1) = \sqrt{3}$.

The equation becomes

$$\begin{aligned} 2x(xv)\left(v + x\frac{dv}{dx}\right) &= x^2 + (xv)^2, \\ 2x^2v\left(v + x\frac{dv}{dx}\right) &= x^2(1+v^2), \\ 2v\left(v + x\frac{dv}{dx}\right) &= 1 + v^2, \\ 2v^2 + 2xv\frac{dv}{dx} &= 1 + v^2, \\ 2xv\,\frac{dv}{dx} &= 1 - v^2, \\ \frac{2v}{1-v^2}\,\frac{dv}{dx} &= \frac{1}{x}, \quad x \neq 0, \ v \neq \pm 1. \end{aligned}$$

We have lost the constant solutions $v(x) = 1$ and $v(x) = -1$ in dividing by $1 - v^2$ in the last step. Neither of these solutions satisfy the initial condition, so we are safe to disregard them.

The equation is now separated, so by our algorithm the solutions are given by

$$\int \frac{2v}{1-v^2}\,dv = \int \frac{1}{x}\,dx$$

Integrating,

$$-\ln(|1 - v^2|) = \ln(|x|) + C \quad \text{or} \quad \ln\left(\left|\frac{1}{1-v^2}\right|\right) = \ln(|x|) + C.$$

Taking exponentials of both sides, we find

$$\frac{1}{1-v^2} = C\,x, \quad \text{so} \quad 1 - v^2 = \frac{C}{x} \quad \text{and} \quad v^2 = 1 - \frac{C}{x}.$$

At this stage, we apply the initial condition $v(1) = \sqrt{3}$ to find that

$$3 = 1 - \frac{C}{1}, \quad \text{so} \quad C = -2.$$

Finally, to convert back to y, we use the definition $v = \frac{y}{x}$:

$$\left(\frac{y}{x}\right)^2 = 1 + \frac{2}{x}, \quad \text{so} \quad y^2 = x^2 + 2x.$$

The conditions $v \neq \pm 1$ on the domain become $y \neq \pm x$.

Solving for y as a function of x, we have two possibilities

$$y(x) = \pm\sqrt{x^2 + 2x}.$$

The branch of the square root that satisfies the initial condition is $y(x) = \sqrt{x^2 + 2x}$.

The function $\sqrt{x^2 + 2x}$ is real-valued when $x^2 + 2x \geq 0$. The interval containing $x = 1$ and satisfing the last inequality is given by $x \geq 0$.

(Here is one way of checking this: completing the square, we have $x^2 + 2x = (x+1)^2 - 1$, so we are looking for x satisfying the condition $(x+1)^2 \geq 1$. This condition holds on the intervals $x + 1 \geq 1$ and $x + 1 \leq -1$. The value $x = 1$ satisfies the first inequality, so the condition is $x + 1 \geq 1$, or equivalently $x \geq 0$.)

The restriction $x \neq 0$ removes the point $x = 0$ from the domain. Finally, we check that $y(x) > \pm x$: this is evident as $y(x) = \sqrt{x^2 + 2x} > \sqrt{x^2} = x$ for $x > 0$.

The solution to the differential equation with the given initial condition is

$$y(x) = \sqrt{x^2 + 2x}, \quad x > 0.$$

2. As we saw in lecture, if the path of a particle described parametrically by $(x(t), y(t))$, $t \in I$ lies on the graph of a function $y(x)$, then by the chain rule

$$\frac{dy}{dt}(t) = \frac{dy}{dx}(x(t))\frac{dx}{dt}(t),$$

so that

$$\frac{dy}{dx}(x(t)) = \frac{\frac{dy}{dt}(t)}{\frac{dx}{dt}(t)} \qquad \text{or, written more compactly,} \qquad \frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} \qquad \text{whenever } \dot{x} \neq 0. \qquad (1)$$

i) The path $t \mapsto (\sqrt{8}\cos(t), \sqrt{2}\sin(t))$, $t \in [0, 2\pi)$ describes an ellipse E. Using (1), find the slope of the tangent line to E at the point $(2, 1)$.

As a reminder, the vector $(\dot{x}(t), \dot{y}(t))$ is called the velocity of the particle at time t.

Suppose that the velocity of a particle is perpendicular to its position for all $t \in I$, and that the path of the particle lies on the graph of the function $y(x)$. Under these hypotheses:

ii) Show that $x(t)\dot{x}(t) + y(t)\dot{y}(t) = 0$ for all $t \in I$.

iii) Show that $y(x)$ is a solution to the differential equation

$$y\frac{dy}{dx} = -x.$$

iv) Conclude that the path of the particle lies on a circle.

Solution. i) The path is at the point $(2,1)$ when

$$\sqrt{8}\cos(t) = 2, \quad \text{so } \cos(t) = \frac{1}{\sqrt{2}} \qquad \text{and} \qquad \sqrt{2}\sin(t) = 1, \quad \text{so } \sin(t) = \frac{1}{\sqrt{2}}.$$

For $t \in [0, 2\pi)$, this holds only when $t = \pi/4$.

Computing the derivatives of the coordinate functions with respect to t, we find

$$\dot{x}(t) = -\sqrt{8}\sin(t), \quad \dot{y}(t) = \sqrt{2}\cos(t),$$

so that by (1)

$$\frac{dy}{dx}(\pi/4) = \frac{\dot{y}(\pi/4)}{\dot{x}(\pi/4)} = \frac{\sqrt{2}\cos(\pi/4)}{-\sqrt{8}\sin(\pi/4)} = \frac{\sqrt{2}/\sqrt{2}}{-\sqrt{8}/\sqrt{2}} = -\frac{1}{2}.$$

Remark. We can verify that this is the slope by finding y as a function of x and computing the derivative directly. The ellipse E has implicit equation

$$\frac{x^2}{8} + \frac{y^2}{2} = 1.$$

Solving for y as a function of x, we have $y(x) = \sqrt{2 - \frac{x^2}{4}}$. (We can check that $y(2) = 1$.) Differentiating, we find

$$y'(x) = \frac{1}{2}\frac{1}{\sqrt{2 - (x^2/4)}}(-2x/4) = -\frac{x}{4\sqrt{2 - (x^2/4)}}.$$

We have $y'(2) = -1/2$.

ii) Because the position $(x(t), y(t))$ and the velocity $(\dot{x}(t), \dot{y}(t))$ are supposed perpendicular for all $t \in I$, their dot product $(x(t), y(t)) \cdot (\dot{x}(t), \dot{y}(t)) = x(t)\dot{x}(t) + y(t)\dot{y}(t)$ is equal to 0 for all $t \in I$.

iii) Rearranging the expression from ii) and applying (1), we get

$$\begin{aligned} y(t)\dot{y}(t) &= -x(t)\dot{x}(t), \\ y(t)\frac{\dot{y}(t)}{\dot{x}(t)} &= -x(t), \\ y(t)\frac{dy}{dx}(x(t)) &= -x(t). \end{aligned}$$

Therefore, the function $y(x)$ satisfies the differential equation

$$y\frac{dy}{dx} = -x.$$

iv) The last equation is separable. Solutions are given implicitly by

$$\int y\,dy = \int -x\,dx.$$

Integrating,

$$y^2 = -x^2 + C,$$

so that

$$x^2 + y^2 = C.$$

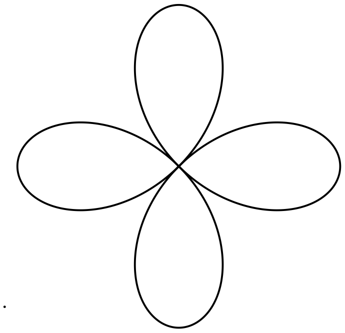
We conclude that the graph of the function $y(x)$, and hence the trajectory of the particle, lies on the circle $x^2 + y^2 = C$, for some constant C.

Optional Problem. For those who enjoy playing around with (algebraic) equations (and can spare a bit of time): Let C be the curve in $\mathbb{R}^2$ defined implicitly by the equation

$$(x^2+y^2)^3 = (x^2-y^2)^2 \quad \text{(the quadrifolium, or four-leafed clover).}$$

Find the points of C where the hypotheses of the Implicit Function Theorem do not hold.

Answer: $(0,0)$, $(1,0)$, $(-1,0)$, $(\sqrt{\frac{2}{27}},\sqrt{\frac{10}{27}})$, $(\sqrt{\frac{2}{27}},-\sqrt{\frac{10}{27}})$, $(-\sqrt{\frac{2}{27}},\sqrt{\frac{10}{27}})$, $(-\sqrt{\frac{2}{27}},-\sqrt{\frac{10}{27}})$.



Solution. Let $F(x,y) = (x^2+y^2)^3-(x^2-y^2)^2$. We are to find the simultaneous solutions to $F(x,y)=0$ and $\frac{\partial F}{\partial y}(x,y)=0$.

Computing the partial derivative:

$$\frac{\partial F}{\partial y}(x,y) = 3(x^2+y^2)^2(2y) - 2(x^2-y^2)(-2y) = 2y\left(3(x^2+y^2)^2-2(x^2-y^2)\right).$$

The partial is zero if and only if $y=0$ or $3(x^2+y^2)^2-2(x^2-y^2)=0$.

In the first case, the points on $F(x,y)=0$ satisfying $y=0$ are solutions to

$$x^6-x^4=0, \quad \text{or} \quad 0 = x^4(x^2-1) = x^4(x-1)(x+1).$$

So, in the first case we obtain the points $(0,0),(1,0)$ and $(-1,0)$.

In the second case, we have $x^2-y^2 = \frac{3}{2}(x^2+y^2)^2$. The points of $F(x,y)=0$ satisfying this equality are solutions to

$$0 = (x^2+y^2)^3 - \frac{9}{4}(x^2+y^2)^4 = (x^2+y^2)^3(1-\frac{9}{4}(x^2+y^2)).$$

The possibility $(x^2+y^2)^3=0$ has solution $(0,0)$, which we already accounted for.

In the second possibility, we have

$$x^2+y^2 = \frac{4}{9}.$$

Together with

$$x^2-y^2 = \frac{3}{2}(x^2+y^2)^2 = \frac{3}{2}\left(\frac{4}{9}\right)^2 = \frac{8}{27},$$

we get a pair of equations we can solve for x^2 and y^2, which yields the remaining four points.