Mthe 237
Lecture 36
Dec. 01, 2017

Topic: Some examples

I. Examples around Newton's Law of Heating/Cooling

Suppose that an object is in contact with a reservoir of heat (often called a "heat bath") that is kept at a constant temperature T_{ext}.

Newton's Law of Heating/Cooling states that the temperature T of the object will tend toward that of the reservoir at a rate proportional to $(T - T_{ext})$.

[This is an approximation that only holds for certain types of heat transfer.]

This leads to the differential equation

$$\dot{T} = -k(T - T_{ext}) \qquad (k > 0).$$

(If $T > T_{ext}$, so that $T - T_{ext} > 0$, $\dot{T}$ should be negative, which explains the sign.)

This differential equation is both linear and separable.

Separating variables,

$$\frac{1}{T - T_{ext}} \frac{dT}{dt} = -k.$$

Integrating both sides,

$$\ln|T - T_{ext}| = -kt + \text{const}.$$

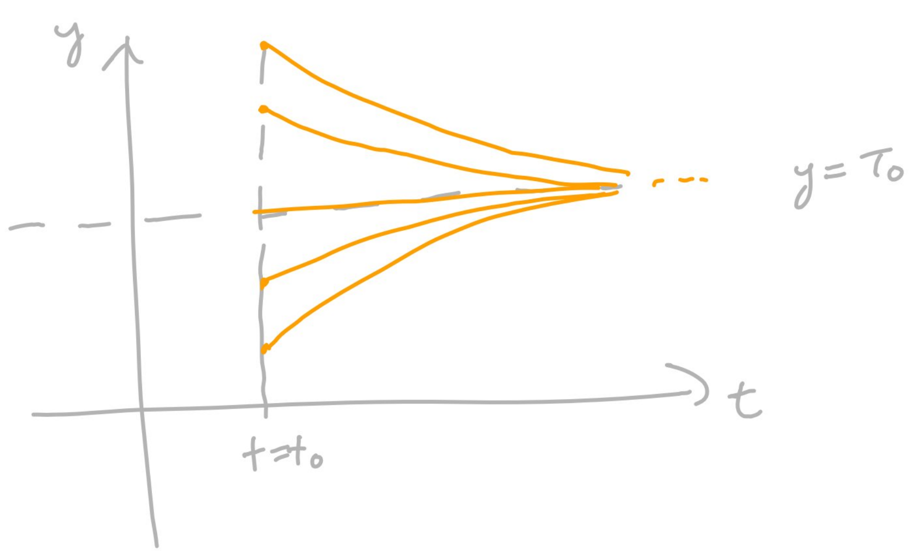
$$T - T_{ext} = A e^{-kt}.$$

If $T(t_0) = T_0$, we can determine A:

$$T_0 - T_{ext} = A e^{-kt_0}$$

$$A = (T_0 - T_{ext}) e^{kt_0}, \quad \text{so that}$$

$$T(t) = T_{ext} + (T_0 - T_{ext}) e^{-k(t-t_0)}$$



The solutions exhibit the exponential decay typical of first-order linear equations, but shifted vertically by T_{ext}.

What if T_{ext} is a function of time?

For instance, T_{ext} could change with the day-night cycle, or yearly, or in a non-periodic manner.

For an interesting example, suppose that

$$T_{ext}(t) = \frac{\cos(\omega t)}{k}, \quad \omega \neq 0 \text{ real.}$$

(Periodic functions can be approximated well by sufficiently many of such terms - This is the theory of <u>Fourier Series</u>.)

By similar reasoning as before, arrive at the differential equation

$$\dot{T} = -k\left(T - \frac{\cos(\omega t)}{k}\right).$$

This is no longer separable, but it <u>is</u> linear.

$$\dot{T} + kT = \cos(\omega t)$$

An annihilator of the right side is

$$p\left(\frac{d}{dt}\right) = \left(\frac{d^2}{dt^2} + \omega^2\right)$$

We have

$$p\left(\frac{d}{dt}\right)x\left(\frac{d}{dt}\right) = \left(\frac{d^2}{dt^2}+w^2\right)\left(\frac{d}{dt}+k\right).$$

Look for a particular solution of the form

$$\varphi_p = A\cos(wt) + B\sin(wt)$$

$$\dot{\varphi}_p = -Aw\sin(wt) + Bw\cos(wt)$$

$$\dot{\varphi}_p + k\varphi_p = (Bw + Ak)\cos(wt) + (Bk - Aw)\sin(wt)$$

The left side should be equal to $k\cos(wt)$,

hence

$$Bw + Ak = 1$$

$$Bk - Aw = 0$$

$$\begin{pmatrix} k & w \\ -w & k \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} A \\ B \end{pmatrix} = \frac{1}{w^2+k^2}\begin{pmatrix} k & -w \\ w & k \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \frac{1}{w^2+k^2}\begin{pmatrix} k \\ w \end{pmatrix}.$$

$$A = k/w^2+k^2$$

$$B = w/w^2+k^2$$

$$\varphi_p = \frac{k}{\omega^2+k^2}\cos(\omega k) + \frac{\omega}{\omega^2+k^2}\sin(\omega k).$$

We can rewrite this in phase-amplitude form as

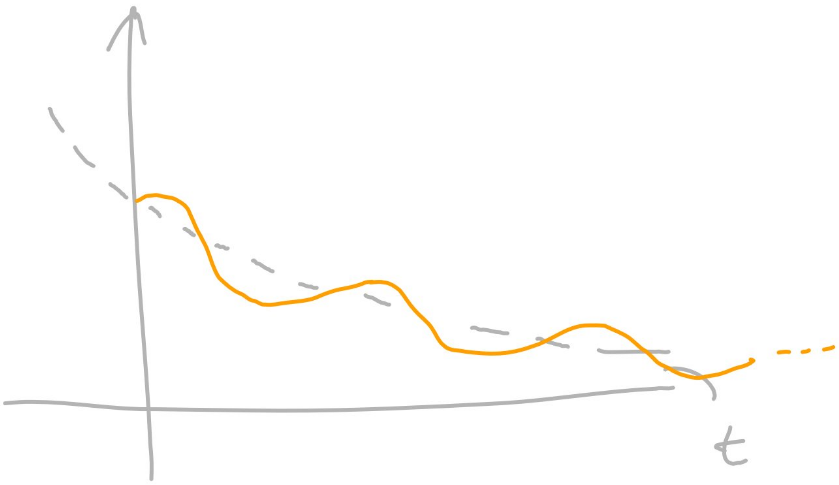
$$\varphi_p = \frac{1}{\sqrt{\omega^2+k^2}}\cos(\omega k + \varphi), \qquad \tan\varphi = -\frac{\omega}{k}.$$

All solutions then have the form

$$\varphi_p + \underbrace{ae^{-kt}}_{\substack{\text{solution of} \\ \frac{dT}{dt}+kT=0}}$$

or

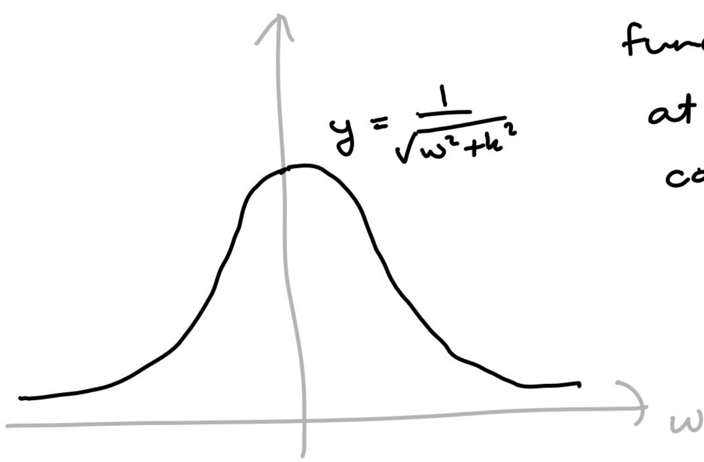
$$\underbrace{\frac{1}{\sqrt{\omega^2+k^2}}\cos(\omega k+\varphi)}_{\substack{\text{periodic} \\ \text{response to} \\ \text{reservoir}}} + \underbrace{ae^{-kt}}_{\substack{\text{transient} \\ \text{term} \\ (\to 0 \text{ as } t\to\infty)}}$$



Solutions oscillate about the graph of ae^{-kt} with the same frequency ω as $\frac{\cos(\omega t)}{k}$, but with a delay of φ.

The amplitude of the response is

$$\frac{1}{\sqrt{\omega^2+k^2}}$$



For a fixed k, this function is maximized at $\omega=0$, and is in this case equal to

$$\frac{1}{\sqrt{0^2+k^2}} = \frac{1}{k},$$

which is equal to the amplitude of

$$\frac{\cos(\omega k)}{k}.$$

There is no resonance here. The response amplitude is bounded above by that of the input.

What happens when the temperature of the reservoir is not controlled, but also follows Newton's Law of Heating / Cooling?

The temperatures T_1, T_2 of the two objects then obey the system of differential equations

$$\dot{T}_1 = -k_1 (T_1 - T_2)$$
$$\dot{T}_2 = -k_2 (T_2 - T_1)$$

$$\begin{pmatrix} \dot{T}_1 \\ \dot{T}_2 \end{pmatrix} = \underbrace{\begin{pmatrix} -k_1 & k_1 \\ k_2 & -k_2 \end{pmatrix}}_{A} \begin{pmatrix} T_1 \\ T_2 \end{pmatrix}$$

Eigenvalues / Eigenvectors of A:

$$\det(A - zI) = \det \begin{pmatrix} -k_1 - z & k_1 \\ k_2 & -k_2 - z \end{pmatrix}$$

$$= (-k_1 - z)(-k_2 - z) - k_1 k_2$$

$$= k_1 k_2 + (k_1 + k_2) z + z^2 - k_1 k_2$$

$$= z \left(z + (k_1 + k_2) \right).$$

The eigenvalues are

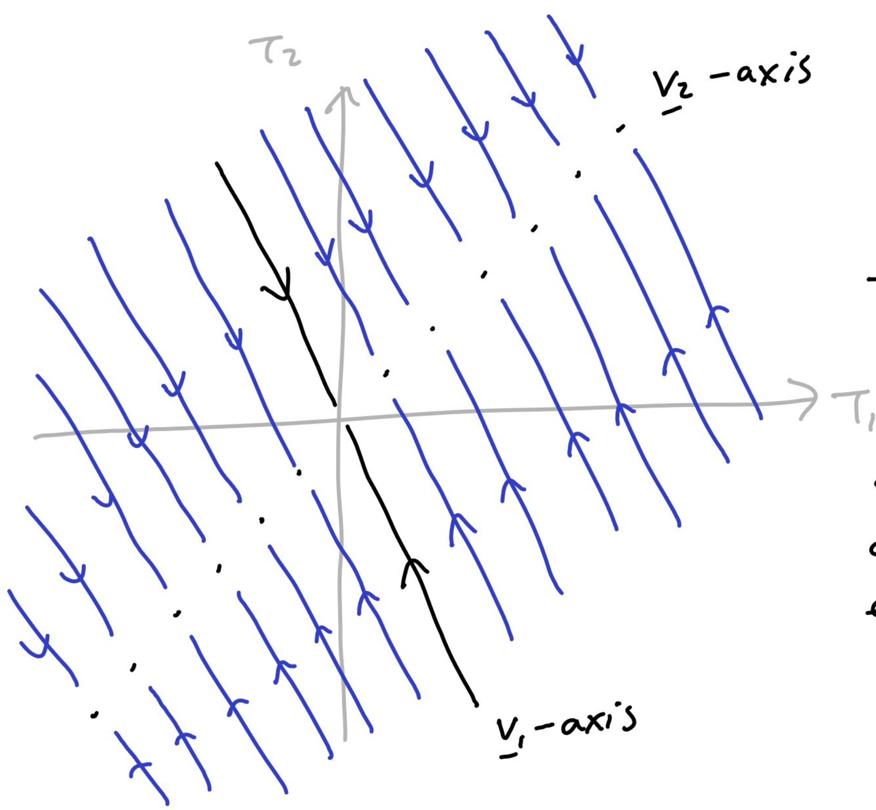
$$\lambda_1 = -(k_1 + k_2) \quad \text{and} \quad \lambda_2 = 0.$$

Eigenvector for λ_1:

$$\ker \begin{pmatrix} k_2 & k_1 \\ k_2 & k_1 \end{pmatrix} \qquad \underline{v_1} = \begin{pmatrix} k_1 \\ -k_2 \end{pmatrix}$$

Eigenvector for λ_2:

$$\ker \begin{pmatrix} -k_1 & k_1 \\ k_2 & -k_2 \end{pmatrix} \qquad \underline{v_2} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$



There are equilibrium points along the $\underline{v_2}$-axis.

These correspond to points at which the two objects have equal temperatures.

In the eigenbasis, the matrix exponential is

$$\begin{pmatrix} e^{-(k_1+k_2)t} & 0 \\ 0 & 1 \end{pmatrix}$$

$$\exp(At) = \begin{pmatrix} k_1 & 1 \\ -k_2 & 1 \end{pmatrix} \begin{pmatrix} e^{-(k_1+k_2)t} & 0 \\ 0 & 1 \end{pmatrix} \frac{1}{k_1+k_2} \begin{pmatrix} 1 & -1 \\ k_2 & k_1 \end{pmatrix}$$

$$= \frac{1}{k_1+k_2} \begin{pmatrix} k_1 & 1 \\ -k_2 & 1 \end{pmatrix} \begin{pmatrix} e^{-(k_1+k_2)t} & -e^{-(k_1+k_2)t} \\ k_2 & k_1 \end{pmatrix}$$

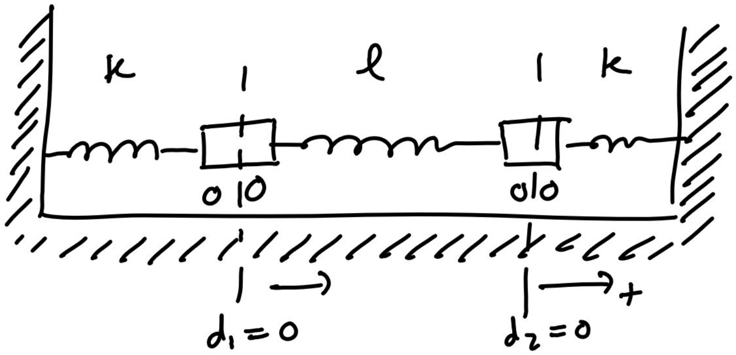
$$= \frac{1}{k_1+k_2} \begin{pmatrix} k_1 e^{-(k_1+k_2)t} + k_2 & -k_1 e^{-(k_1+k_2)t} + k_1 \\ -k_2 e^{-(k_1+k_2)t} + k_2 & k_2 e^{-(k_1+k_2)t} + k_1 \end{pmatrix}$$

So that the solution satisfying $\underline{x}(0) = \begin{pmatrix} T_1(0) \\ T_2(0) \end{pmatrix}$

is $\exp(tA) \begin{pmatrix} T_1(0) \\ T_2(0) \end{pmatrix}$

$$= \frac{1}{k_1+k_2} \begin{pmatrix} k_1 \left(T_1(0) - T_2(0)\right) e^{-(k_1+k_2)t} + \left(k_1 T_2(0) + k_2 T_1(0)\right) \\ k_2 \left(T_2(0) - T_1(0)\right) e^{-(k_1+k_2)t} + \left(k_2 T_1(0) + k_1 T_2(0)\right) \end{pmatrix}.$$

II.



Assume $m_1 = m_2 = 1$.

Previously, we found that the displacements of the two masses from their rest positions satisfy the system of equations

$$\ddot{d}_1 = -(k+l)d_1 + l d_2$$
$$\ddot{d}_2 = l d_1 - (k+l) d_2$$

This has the equivalent first-order system

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -(k+l)x_1 + l x_3$$
$$\dot{x}_3 = x_4$$
$$\dot{x}_4 = l x_1 - (k+l) x_3$$

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -k-l & 0 & l & 0 \\ 0 & 0 & 0 & 1 \\ l & 0 & -k-l & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

Skipping the (quite long) details, we find that the eigenvalues of A are

$$\underbrace{i\sqrt{k},\ -i\sqrt{k}}_{\text{Eigenvector}},\quad \underbrace{i\sqrt{k+2l},\ -i\sqrt{k+2l}}_{\text{Eigenvector}}$$

Eigenvector $\begin{pmatrix} 1 \\ -i\sqrt{k} \\ 1 \\ -i\sqrt{k} \end{pmatrix}$ $\qquad$ Eigenvector $\begin{pmatrix} -1 \\ i\sqrt{k+2l} \\ 1 \\ -i\sqrt{k+2l} \end{pmatrix}$

real: $\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ $\quad$ imaginary: $\begin{pmatrix} 0 \\ -\sqrt{k} \\ 0 \\ -\sqrt{k} \end{pmatrix}$ $\qquad$ real: $\begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ $\quad$ imaginary: $\begin{pmatrix} 0 \\ \sqrt{k+2l} \\ 0 \\ -\sqrt{k+2l} \end{pmatrix}$

In the basis of real/imaginary parts of the two complex eigenvectors, the matrix exponential becomes

$$\left(\begin{array}{cc|cc} \cos(\sqrt{k}t) & \sin(\sqrt{k}t) & 0 & 0 \\ -\sin(\sqrt{k}t) & \cos(\sqrt{k}t) & 0 & 0 \\ \hline 0 & 0 & \cos(\sqrt{k+2l}\,t) & \sin(\sqrt{k+2l}\,t) \\ 0 & 0 & -\sin(\sqrt{k+2l}\,t) & \cos(\sqrt{k+2l}\,t) \end{array}\right)$$

We can interpret the variables x_1, x_2, x_3, x_4 as:

x_1 - displacement of first mass from rest

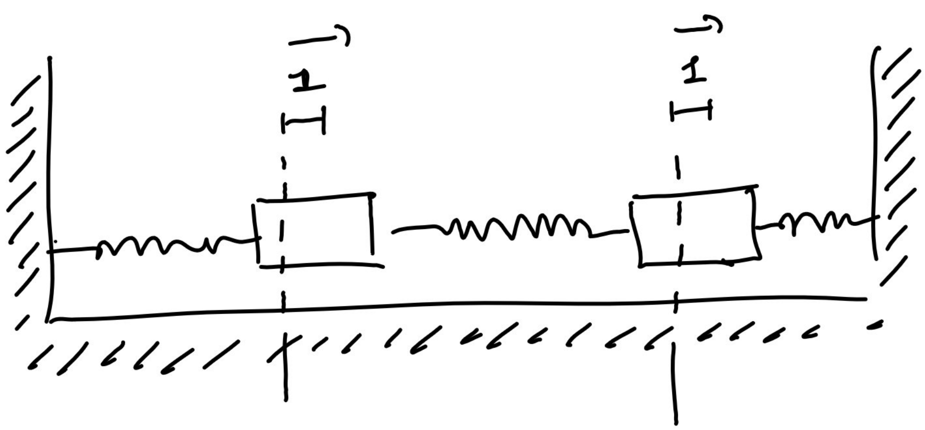
x_2 - velocity of first mass

x_3 - displacement of second mass from rest

x_4 - velocity of second mass

Taking the eigenvector $\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ as the initial condition,

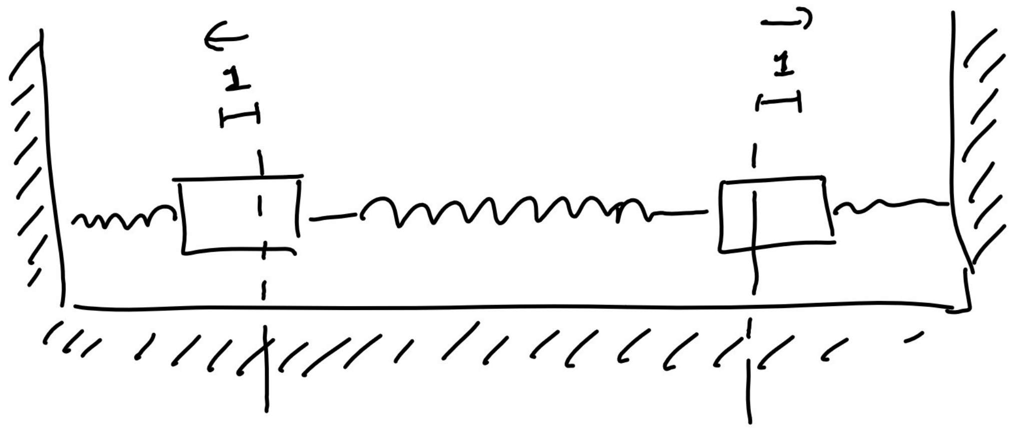
$$\underline{x}(t) = \begin{pmatrix} \cos(\sqrt{k}t) \\ -\sqrt{k}\sin(\sqrt{k}t) \\ \cos(\sqrt{k}t) \\ -\sqrt{k}\sin(\sqrt{k}t) \end{pmatrix}$$



Both masses are displaced one unit to the right. The resulting motion does not compress or stretch the middle spring. The two masses oscillate with frequency $\sqrt{k}$, the natural frequency of the left and right springs. The masses move "in phase".

Taking the eigenvector $\begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ as the initial condition,

$$\underline{x}(t) = \begin{pmatrix} -\cos(\sqrt{k+2\ell}\,t) \\ \sqrt{k+2\ell}\,\sin(\sqrt{k+2\ell}\,t) \\ \cos(\sqrt{k+2\ell}\,t) \\ -\sqrt{k+2\ell}\,\sin(\sqrt{k+2\ell}\,t) \end{pmatrix}$$



The left mass is displaced one unit left, the right mass is displaced one unit right.

The middle spring is initially stretched by 2 units.

The masses move "inward", then "outward", completely "out of phase" with one another, both at frequency $\sqrt{k+2\ell}$.

The general motion starting at rest is a superposition of these two motions, since

$$\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \text{ span } \begin{pmatrix} x_1 \\ 0 \\ x_3 \\ 0 \end{pmatrix}.$$

The other two eigenvectors correspond to starting the masses at rest, but with prescribed initial velocities.

Any motion (with arbitrary initial positions and velocities) is a superposition of the four eigenvectors.

— The End —

Thanks for a fun semester, everyone.