Mthe 237
Lecture 35
Nov. 29, 2017

Topics:
- Time-advance map
- Variation of Parameters for First-Order Systems

Consider the system of differential equations

$$\dot{\underline{x}} = A(t)\underline{x}, \quad A(t) \text{ continuous over } I.$$

Last time, we discussed the maps

$$\Psi_{t_0} : \left\{ \begin{array}{c} \text{solutions of} \\ \dot{\underline{x}} = A(t)\underline{x} \end{array} \right\} \longrightarrow \mathbb{R}^r$$
$$\varphi \longmapsto \varphi(t_0)$$

that are defined for all $t_0 \in I$.

We showed previously (as a consequence of the Existence and Uniqueness Theorem) that for each $t_0 \in I$, Ψ_{t_0} is an isomorphism of vector spaces.

Def. Let $t_0, t_1 \in I$. The map

$$g_{t_0}^{t_1} = \Psi_{t_1} \circ \Psi_{t_0}^{-1} : \mathbb{R}^r \longrightarrow \mathbb{R}^r$$

is called the <u>time-transition map from t_0 to t_1 for the system $\dot{\underline{x}} = A(t)\underline{x}$</u>.

The maps $g_{t_0}^{t_1}$ are vector space isomorphisms.

Notation: In control theory, these maps are denoted by $\Phi(t_1, t_0)$ and called state-transition maps. The notation $g_{t_0}^{t_1}$ comes from the branch of pure mathematics called dynamical systems.

Example:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}_{A} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\exp(A(t-t_0)) = \begin{pmatrix} \cos(t-t_0) & \sin(t-t_0) \\ -\sin(t-t_0) & \cos(t-t_0) \end{pmatrix}.$$

The solution with $\underline{x}(t_0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is

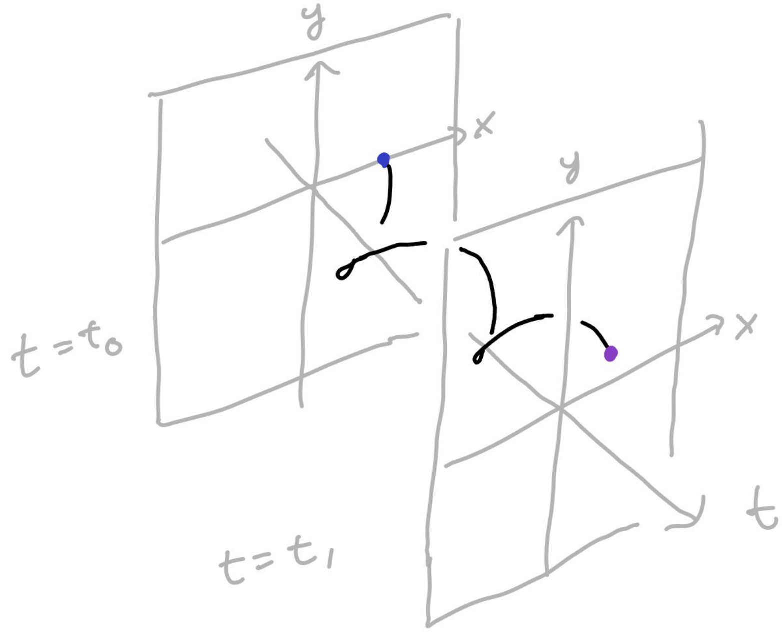
$$\varphi : t \longmapsto \begin{pmatrix} \cos(t-t_0) \\ -\sin(t-t_0) \end{pmatrix}$$

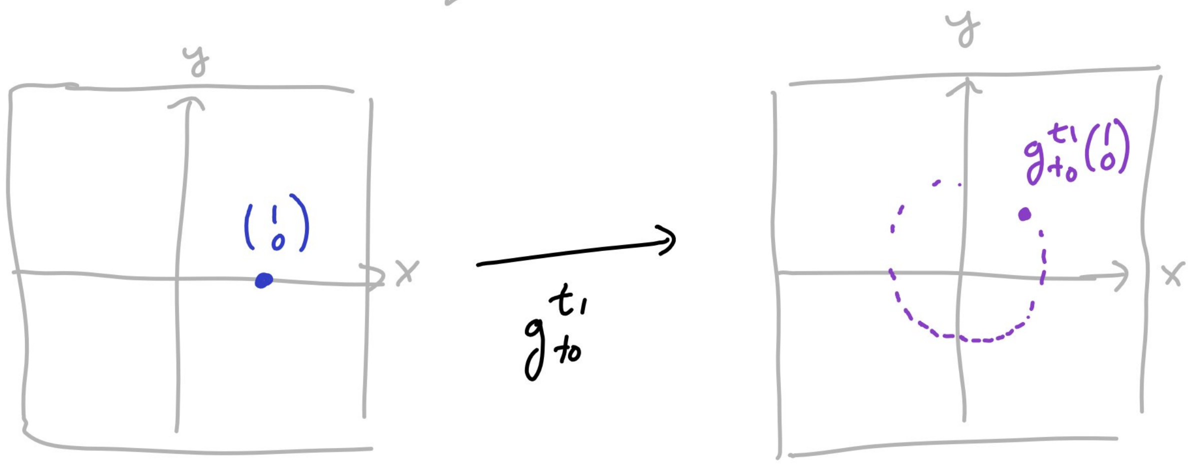
Thus, $\Psi_{t_0}^{-1}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \varphi$

and

$$g_{t_0}^{t_1}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \Psi_{t_1}(\varphi) = \begin{pmatrix} \cos(t_1-t_0) \\ -\sin(t_1-t_0) \end{pmatrix}$$

The map $g_{t_0}^{t_1}$ applied to $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ "finds the solution passing through $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ at t_0" and "sees what point that solution passes through at t_1".





§ Properties of $g_{t_0}^{t_1}$

Theorem

(a) For any $t_0, t_1, t_2 \in I$,

$$g_{t_1}^{t_2} \, g_{t_0}^{t_1} = g_{t_0}^{t_2}.$$

(b) For any $t_0, t_1 \in I$,

$$g_{t_1}^{t_0} \, g_{t_0}^{t_1} = \text{Identity map on } \mathbb{R}^r$$

(c) The solution to the system of differential equations

$$\dot{\underline{x}} = A(t)\underline{x}, \qquad \underline{x}(t_0) = \underline{x}_0$$

is given by

$$\underline{x} : t \longmapsto g_{t_0}^{t}\,\underline{x}_0$$

(d) If $t \mapsto A(t)$ is a constant function, then

$$g_{t_0}^{t} = \exp(A(t-t_0)) \qquad \text{for all } t \in I. \text{ and any } t_0 \in I$$

(e) For any $t_0 \in I$, the function

$$t \longmapsto g_{t_0}^{t}$$

is a differentiable (matrix-valued) function. Its derivative with respect to t is denoted by $\dot{g}_{t_0}^{t}$.

$\dot{g}_{t_0}^{t}$ satisfies $\dot{g}_{t_0}^{t} = A(t)\, g_{t_0}^{t}$.

Proof:

(a) $$g_{t_1}^{t_2}\, g_{t_0}^{t_1} = \left(\Psi_{t_2} \circ \Psi_{t_1}^{-1}\right) \circ \left(\Psi_{t_1} \circ \Psi_{t_0}^{-1}\right)$$

$$= \Psi_{t_2} \circ \Psi_{t_0}^{-1} = g_{t_0}^{t_2}.$$

(b) $$g_{t_1}^{t_0}\, g_{t_0}^{t_1} = \left(\Psi_{t_0} \circ \Psi_{t_1}^{-1}\right) \circ \left(\Psi_{t_1} \circ \Psi_{t_0}^{-1}\right)$$

$$= \text{Identity map on } \mathbb{R}^r.$$

(c) The solution of

$$\dot{\underline{x}} = A(t)\,\underline{x}$$

with $\underline{x}(t_0) = \underline{x}_0$ is $\Psi_{t_0}^{-1}(\underline{x}_0) = \varphi$.

Then, for any $t \in I$,

$$\varphi(t) = \Psi_t(\varphi) = \Psi_t\left(\Psi_{t_0}^{-1}(\underline{x}_0)\right)$$

$$= \left(\Psi_t \circ \Psi_{t_0}^{-1}\right)(\underline{x}_0)$$

$$= g_{t_0}^{t}\, \underline{x}_0.$$

Therefore, the solution may be written as

$$t \longmapsto g_{t_0}^{t}\, \underline{x}_0, \quad t \in I.$$

(d) For any $t \in I$, $t_0 \in I$, the maps

$$\exp(A(t-t_0)) \quad \text{and} \quad g_{t_0}^{t}$$

are linear maps $\mathbb{R}^r \to \mathbb{R}^r$. Let $(\underline{e}_1, \dots, \underline{e}_r)$ be a basis of $\mathbb{R}^r$.

To show that $\exp(A(t-t_0)) = g_{t_0}^t$, it is enough to show that

$$\exp(A(t-t_0))\,\underline{e}_j = g_{t_0}^t\,\underline{e}_j \quad \text{for all } j=1,\dots,r$$

$\left(\begin{array}{l} \text{By linearity, it then would follow that} \\ \exp(A(t-t_0))\,\underline{v} = g_{t_0}^t\,\underline{v} \text{ for all } \underline{v}\in\mathbb{R}^r. \end{array}\right)$

But, $t \mapsto \exp(A(t-t_0))\underline{e}_j$ and $t \mapsto g_{t_0}^t \underline{e}_j$ are both solutions of the system of differential equations

$$\dot{\underline{x}} = A(t)\,\underline{x}, \qquad \underline{x}(t_0) = \underline{e}_j.$$

Therefore, by the uniqueness theorem,

$$\exp(A(t-t_0))\,\underline{e}_j = g_{t_0}^t\,\underline{e}_j \quad \text{for all } t\in I.$$

The conclusion follows.

(e) Fix a basis $(\underline{e}_1, \dots, \underline{e}_r)$ of $\mathbb{R}^r$.

Let $\varphi_j = \Psi_{t_0}^{-1}(\underline{e}_j)$. Then for any $t\in I$,

$$\left[g_{t_0}^t\right]_{(\underline{e}_1,\dots,\underline{e}_r)}^{(\underline{e}_1,\dots,\underline{e}_r)} = \begin{pmatrix} | & & | \\ \varphi_1(t) & \cdots & \varphi_r(t) \\ | & & | \end{pmatrix}$$

Since φ_j are solutions of a system of differential equations, they are differentiable functions of t. Therefore, $t \mapsto g_{t_0}^{t}$ is also a differentiable function of t.

By definition,

$$\dot{g}_{t_0}^{t} = \begin{pmatrix} | & & | \\ \dot{\varphi}_1 & \cdots & \dot{\varphi}_r \\ | & & | \end{pmatrix}$$

$$= \begin{pmatrix} | & & | \\ A(t)\varphi_1 & \cdots & A(t)\varphi_r \\ | & & | \end{pmatrix}$$

$$= A(t) \begin{pmatrix} | & & | \\ \varphi_1 & \cdots & \varphi_r \\ | & & | \end{pmatrix}$$

$$= A(t)\, g_{t_0}^{t}$$

$\square$

§ <u>Computing</u> $g_{t_0}^{t_1}$

In the case $\dot{\underline{x}} = A\underline{x}$, where A does not depend on t, the previous proposition shows that

$$g_{t_0}^{t_1} = \exp(A(t_1 - t_0)).$$

In general, to compute $g_{t_0}^{t_1}$ one needs to be able to explicitly find solutions of

$$\underline{\dot{x}} = A(t)\,\underline{x}.$$

Then, for any basis $(\underline{e}_1, \dots, \underline{e}_r)$ of $\mathbb{R}^r$, let

$\varphi_j = \Psi_{t_0}^{-1}(\underline{e}_j)$ (φ_j is the solution passing through $\underline{e}_j$ at time t_0.)

write out

$$\varphi_j(t) = \begin{pmatrix} \varphi_{1j}(t) \\ \vdots \\ \varphi_{rj}(t) \end{pmatrix}, \quad t \in I, \quad \varphi_{ij} : I \to \mathbb{R}$$

in the basis $(\underline{e}_1, \dots, \underline{e}_r)$.

The matrix of $g_{t_0}^{t_1}$ with respect to the basis $(\underline{e}_1, \dots, \underline{e}_r)$ is then

$$\left[g_{t_0}^{t_1} \right]_{(\underline{e}_1, \dots, \underline{e}_r)}^{(\underline{e}_1, \dots, \underline{e}_r)} = \begin{pmatrix} | & & | \\ \varphi_1(t_1) & \cdots & \varphi_r(t_1) \\ | & & | \end{pmatrix}.$$

<u>Example</u>: This is a contrived example to illustrate the previous procedure. In general, it is hard to solve $\underline{\dot{x}} = A(t)\,\underline{x}$.

Let $\underline{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\underline{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Suppose in basis $(\underline{e}_1, \underline{e}_2)$, the system of equations $\underline{\dot{x}} = A\,\underline{x}$ becomes

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} t & 0 \\ 0 & \frac{1}{t} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \qquad t > 0.$$

$\dot{x} = tx$. Separate variables

$$\frac{1}{x}\frac{dx}{dt} = t.$$

Integrate

$$\ln|x| = t^2/2 + \text{const.}$$

$$x(t) = Ae^{t^2/2}.$$

$\dot{y} = \frac{1}{t}y$. Separate variables

$$\frac{1}{y}\frac{dy}{dt} = \frac{1}{t}.$$

$$\ln|y| = \ln|t|$$

$$y = Bt.$$

The solution satisfying $\underline{x}(t_0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \underline{e}_1$

will have $\begin{pmatrix} Ae^{t_0^2/2} \\ Bt_0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow \begin{matrix} A = e^{-t_0^2/2} \\ B = 0 \end{matrix}$

$$\varphi_1 : t \longmapsto \begin{pmatrix} \exp\left(\frac{t^2 - t_0^2}{2}\right) \\ 0 \end{pmatrix}.$$

The solution satisfying $\underline{x}(t_0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \underline{e}_2$

will have $\begin{pmatrix} Ae^{t_0^2/2} \\ Bt_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Rightarrow \begin{matrix} A = 0 \\ B = \frac{1}{t_0} \end{matrix}$

$$\varphi_2 : t \longmapsto \begin{pmatrix} 0 \\ \frac{t}{t_0} \end{pmatrix}$$

Then, $$\left[g_{t_0}^{t_1} \right]_{(\underline{e}_1, \underline{e}_2)}^{(\underline{e}_1, \underline{e}_2)} = \begin{pmatrix} | & | \\ \varphi_1(t_1) & \varphi_2(t_1) \\ | & | \end{pmatrix}$$

$$= \begin{pmatrix} \exp\left(\frac{t_1^2 - t_0^2}{2}\right) & 0 \\ 0 & \frac{t_1}{t_0} \end{pmatrix}.$$

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§ Variation of Parameters

This is a technique for solving nonhomogeneous linear systems of differential equations.

$$\dot{\underline{x}} = A(t)\underline{x} + \underline{F}(t).$$

Motivation: Solution of the associated homogeneous system $\dot{\underline{x}} = A(t)\underline{x}$, $\underline{x}(t_0) = \underline{x}_0$ is

$$t \longmapsto g_{t_0}^{t} \underline{x}_0.$$

Try to look for a solution of

$$\dot{\underline{x}} = A(t)\underline{x} + \underline{F}(t)$$

of the form

$$\underline{x} : t \longmapsto g_{t_0}^{t} \underline{u}(t).$$

Differentiating with respect to t,

$$\dot{\underline{x}}(t) = \dot{g}^{t}_{t_0}\, \underline{u}(t) + g^{t}_{t_0}\, \dot{\underline{u}}(t)$$

$$= A(t)\, g^{t}_{t_0}\, \underline{u}(t) + g^{t}_{t_0}\, \dot{\underline{u}}(t)$$

$$= A(t)\, \underline{x}(t) + g^{t}_{t_0}\, \dot{\underline{u}}(t).$$

We see that $\underline{x}$ is a solution of

$\dot{\underline{x}} = A(t)\underline{x} + \underline{F}(t)$ if and only if

$g^{t}_{t_0}\, \dot{\underline{u}}(t) = \underline{F}(t)$ for all $t \in I$

So that we require that

$$\dot{\underline{u}}(t) = \left(g^{t}_{t_0}\right)^{-1} \underline{F}(t)$$

$$= g^{t_0}_{t}\, \underline{F}(t)$$

and

$$\underline{u}(t) = \underline{u}(t_0) + \int_{t_0}^{t} g^{t_0}_{\tau}\, \underline{F}(\tau)\, d\tau.$$

Finally,

$$\underline{x}(t) = g^{t}_{t_0}\, \underline{u}(t) = g^{t}_{t_0}\, \underline{u}(t_0) + g^{t}_{t_0} \int_{t_0}^{t} g^{t_0}_{\tau}\, \underline{F}(\tau)\, d\tau$$

$$= g^{t}_{t_0}\, \underline{u}(t_0) + \int_{t_0}^{t} g^{t}_{\tau}\, \underline{F}(\tau)\, d\tau$$

We have

$$\underline{x}(t_0) = g_{t_0}^{t_0}\, \underline{u}(t_0) + 0, \quad \text{so} \quad \underline{u}(t_0) = \underline{x}(t_0).$$

<u>Theorem</u>.

The solution of

$$\dot{\underline{x}} = A(t)\,\underline{x} + \underline{F}(t), \qquad \underline{x}(t_0) = \underline{x}_0$$

is

$$\underline{x} : t \mapsto g_{t_0}^{t}\, \underline{x}_0 + \int_{t_0}^{t} g_{\tau}^{t}\, \underline{F}(\tau)\, d\tau.$$

In the special case when $A(t)$ is constant with t,

the solution of

$$\dot{\underline{x}} = A\underline{x} + \underline{F}(t), \qquad \underline{x}(t_0) = \underline{x}_0$$

is

$$\underline{x} : t \longmapsto \exp\big(A(t-t_0)\big)\, \underline{x}_0 + \int_{t_0}^{t} \exp\big(A(t-\tau)\big)\, \underline{F}(\tau)\, d\tau.$$

Example.

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} t \\ t^2 \end{pmatrix}, \qquad \underline{x}(t_0) = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$$

In this example, x and y are "unlinked", but let's solve it using the variation of parameters method we just developed.

$$\exp(A(t-t_0)) = \begin{pmatrix} e^{2(t-t_0)} & 0 \\ 0 & e^{3(t-t_0)} \end{pmatrix}$$

$$\exp(A(t-\tau)) \begin{pmatrix} \tau \\ \tau^2 \end{pmatrix} = \begin{pmatrix} e^{2(t-\tau)} & 0 \\ 0 & e^{3(t-\tau)} \end{pmatrix} \begin{pmatrix} \tau \\ \tau^2 \end{pmatrix}$$

$$= \begin{pmatrix} \tau e^{2(t-\tau)} \\ \tau^2 e^{3(t-\tau)} \end{pmatrix}$$

$$\int_{t_0}^{t} \exp(A(t-\tau)) \begin{pmatrix} \tau \\ \tau^2 \end{pmatrix} d\tau$$

$$= \begin{pmatrix} \int_{t_0}^{t} \tau e^{2(t-\tau)} \, d\tau \\ \int_{t_0}^{t} \tau^2 e^{3(t-\tau)} \, d\tau \end{pmatrix}$$

$$\int_{t_0}^{t} \underbrace{\tau}_{u}\, \underbrace{e^{2(t-\tau)}}_{v'}\, d\tau = -\frac{1}{2}\tau e^{2(t-\tau)}\Big|_{t_0}^{t} - \int_{t_0}^{t}\left(-\frac{1}{2}\right)e^{2(t-\tau)}\,d\tau$$

$$= -\frac{1}{2}t + \frac{1}{2}t_0 e^{2(t-t_0)} - \frac{1}{4}e^{2(t-\tau)}\Big|_{t_0}^{t}$$

$$= -\frac{1}{2}t + \frac{1}{2}t_0 e^{2(t-t_0)} - \frac{1}{4} + \frac{1}{4}e^{2(t-t_0)}$$

$$= \frac{2t_0+1}{4}e^{2(t-t_0)} - \frac{2t+1}{4}$$

$$\int_{t_0}^{t} \underbrace{\tau^2}_{u}\, \underbrace{e^{3(t-\tau)}}_{v'}\, d\tau = -\frac{1}{3}\tau^2 e^{3(t-\tau)}\Big|_{t_0}^{t} - \int_{t_0}^{t}\left(-\frac{2}{3}\right)\tau e^{3(t-\tau)}\,d\tau$$

$$= -\frac{1}{3}t^2 + \frac{1}{3}t_0^2 e^{3(t-t_0)} + \frac{2}{3}\int_{t_0}^{t} \underbrace{\tau}_{u}\, \underbrace{e^{3(t-\tau)}}_{v'}\, d\tau$$

$$\int_{t_0}^{t} \tau e^{3(t-\tau)}\,d\tau = -\frac{1}{3}\tau e^{3(t-\tau)}\Big|_{t_0}^{t} - \int_{t_0}^{t}\left(-\frac{1}{3}\right)e^{3(t-\tau)}\,d\tau$$

$$= -\frac{1}{3}t + \frac{1}{3}t_0 e^{3(t-t_0)} - \frac{1}{9}e^{3(t-\tau)}\Big|_{t_0}^{t}$$

So that

$$\int_{t_0}^{t} \tau^2 e^{3(t-\tau)}\, d\tau = -\frac{1}{3}t^2 + \frac{1}{3}t_0^2 e^{3(t-t_0)} - \frac{2}{9}t + \frac{2}{9}t_0 e^{3(t-t_0)} - \frac{2}{27} + \frac{2}{27}e^{3(t-t_0)}$$

$$= \frac{9t_0^2 + 6t_0 + 2}{27} e^{3(t-t_0)} - \frac{9t^2 + 6t + 2}{27}$$

Finally, the solution is

$$\underline{x}(t) = \exp\big(A(t-t_0)\big)\,\underline{x}_0 + \int_{t_0}^{t} \exp\big(A(t-\tau)\big)\,\underline{F}(\tau)\, d\tau$$

$$= \begin{pmatrix} 4e^{2(t-t_0)} \\ 5e^{3(t-t_0)} \end{pmatrix} + \begin{pmatrix} \frac{2t_0+1}{4} e^{2(t-t_0)} - \frac{2t+1}{4} \\ \frac{9t_0^2+6t_0+2}{27} e^{3(t-t_0)} - \frac{9t^2+6t+2}{27} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{2t_0+17}{4} e^{2(t-t_0)} - \frac{2t+1}{4} \\ \frac{9t_0^2+6t_0+137}{27} e^{3(t-t_0)} - \frac{9t^2+6t+2}{27} \end{pmatrix}.$$