MTHE 237 | Topics: • Fundamental Matrices
Lecture 34 | • Wronskians for First-Order Systems
Nov. 28, 2017

Today, we'll work with the system of equations

$$\dot{\underline{x}} = A(t)\,\underline{x}\,, \qquad \underline{x}(0) = \underline{x}_0 \in \mathbb{R}^r\,, \qquad (*)$$

where $A(t)$ is a continuous (matrix-valued) function over $I \subset \mathbb{R}$.

When discussing the vector space structure of the set of solutions of system $(*)$, we introduced the maps

$$\Psi_{t_0} : \left\{ \begin{array}{c} \text{Solutions of} \\ \dot{\underline{x}} = A(t)\,\underline{x} \end{array} \right\} \longrightarrow \mathbb{R}^r, \qquad \varphi \longmapsto \varphi(t_0)$$

where t_0 is any point of I,

and showed that for each $t_0 \in I$, Ψ_{t_0} is an isomorphism of vector spaces.

We can think of Ψ_{t_0} as evaluating the solutions at a fixed point t_0.

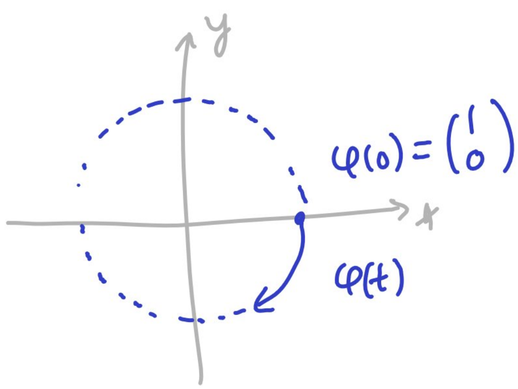
Example. The solution of

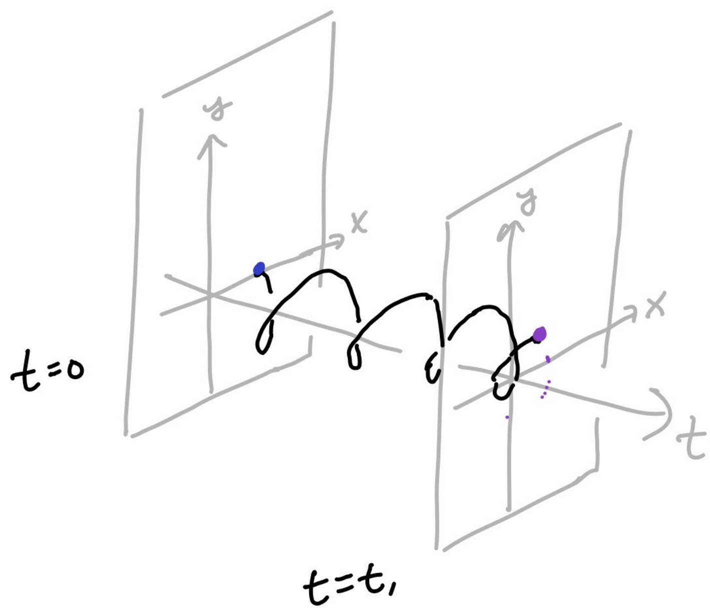
$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

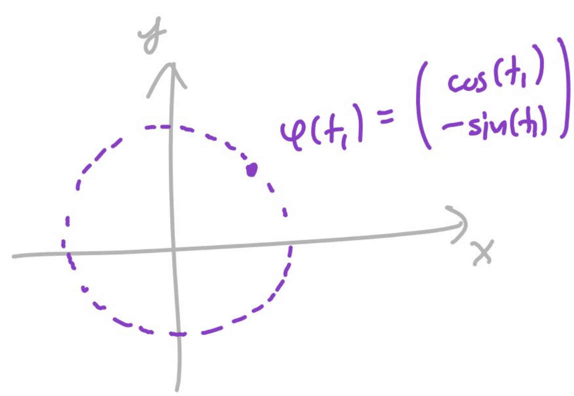
satisfying $\begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is $\varphi(t) = \begin{pmatrix} \cos(t) \\ -\sin(t) \end{pmatrix}$.

We have

$$\Psi_{t_1}(\varphi) = \varphi(t_1) = \begin{pmatrix} \cos(t_1) \\ -\sin(t_1) \end{pmatrix}$$







Reminder: Let V and W be finite-dimensional vector spaces.

Let $A: V \to W$ be a linear map.

That is, A is a map of sets, such that

$$A(v+v') = A(v) + A(v') \quad \text{for any } v, v' \in V$$

and $$A(av) = a A(v) \quad \text{for any } v \in V,\ a \in \mathbb{R}.$$

Suppose that $\beta = (\underline{e}_1, \dots, \underline{e}_r)$ is a basis of V and
$\gamma = (\underline{f}_1, \dots, \underline{f}_s)$ is a basis of W.

Then the matrix of A with respect to the pair of basis β, γ is defined to be

$$[A]_\beta^\gamma = \begin{pmatrix} | & & | \\ A\underline{e}_1 & \dots & A\underline{e}_r \\ | & & | \end{pmatrix}$$

Because order of the elements matters when talking about bases, write $(\underline{e}_1, \dots, \underline{e}_r)$ instead of $\{\underline{e}_1, \dots, \underline{e}_r\}$

For <u>example</u>: Suppose that

$$A\underline{e}_1 = \underline{f}_1 + 2\underline{f}_2$$

$$A\underline{e}_2 = 3\underline{f}_1 + 4\underline{f}_2$$

$$A\underline{e}_3 = 5\underline{f}_1 + 6\underline{f}_2$$

Then $$[A]_{(\underline{e}_1, \underline{e}_2, \underline{e}_3)}^{(\underline{f}_1, \underline{f}_2)} = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix}.$$

With this definition of $[A]_\beta^\gamma$,

$$[A]_\beta^\gamma [v]_\beta = [w]_\gamma, \quad \text{where} \quad w = Av.$$

For <u>example</u>: Continue with the previous A. By linearity,

$$\begin{aligned} A(a\underline{e}_1 + b\underline{e}_2 + c\underline{e}_3) &= aA\underline{e}_1 + bA\underline{e}_2 + cA\underline{e}_3 \\ &= a(\underline{f}_1 + 2\underline{f}_2) + b(3\underline{f}_1 + 4\underline{f}_2) + c(5\underline{f}_1 + 6\underline{f}_2) \\ &= (a + 3b + 5c)\underline{f}_1 + (2a + 4b + 6c)\underline{f}_2 \end{aligned}$$

so that if $v = a\underline{e}_1 + b\underline{e}_2 + c\underline{e}_3$

$$[v]_{(\underline{e}_1, \underline{e}_2, \underline{e}_3)} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$w = (a + 3b + 5c)\underline{f}_1 + (2a + 4b + 6c)\underline{f}_2$$

$$[w]_{(\underline{f}_1, \underline{f}_2)} = \begin{pmatrix} a + 3b + 5c \\ 2a + 4b + 6c \end{pmatrix}$$

We computed above that $Av = w$.

On the other hand,

$$\begin{aligned} [A]_{(\underline{e}_1, \underline{e}_2, \underline{e}_3)}^{(\underline{f}_1, \underline{f}_2)} [v]_{(\underline{e}_1, \underline{e}_2, \underline{e}_3)} &= \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \\ &= \begin{pmatrix} a + 3b + 5c \\ 2a + 4b + 6c \end{pmatrix} = [w]_{(\underline{f}_1, \underline{f}_2)}. \end{aligned}$$

Def. A basis for the vector space $\left\{ \begin{array}{c} \text{solutions of} \\ \dot{\underline{x}} = A(t)\,\underline{x} \quad (*) \end{array} \right\}$ is called a <u>fundamental set</u> of <u>solutions</u> of <u>$(*)$</u>.

<u>Observation</u>: Since $\Psi_{t_0} : \left\{ \begin{array}{c} \text{Solutions of} \\ \dot{\underline{x}} = A(t)\,\underline{x} \end{array} \right\} \to \mathbb{R}^r$

is an isomorphism of vector spaces,

for any basis $(\underline{e}_1, \ldots, \underline{e}_r)$ of $\mathbb{R}^r$, the inverse images $\left(\Psi_{t_0}^{-1}(\underline{e}_1), \ldots, \Psi_{t_0}^{-1}(\underline{e}_r)\right)$ form a fundamental set of solutions.

In the case when $A(t) = A$ for all t (i.e. A is a constant matrix that does not depend on t), the solution of

$$\dot{\underline{x}} = A\,\underline{x}$$

satisfying $\underline{x}(t_0) = \underline{x}_0$ is given by

$$\underline{x}(t) = \exp\left(A(t-t_0)\right)\underline{x}_0.$$

In other words,

$$\Psi_{t_0}^{-1}(\underline{x}_0) = \exp\left(A(t-t_0)\right)\underline{x}_0.$$

Taking $\underline{e}_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \ldots, \underline{e}_r = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$

(this is referred to as the standard basis of $\mathbb{R}^r$), and

$$\varphi_1 = \exp(A(t-t_0))\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \ldots, \varphi_r = \exp(A(t-t_0))\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix},$$

it follows that $(\varphi_1, \ldots, \varphi_r)$ is a basis for the vector space $\left\{ \begin{array}{c} \text{Solutions of} \\ \dot{\underline{x}} = A\underline{x} \end{array} \right\}$.

<u>Example</u>:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}_{A} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\exp(A(t-t_0)) = \begin{pmatrix} \cos(t-t_0) & \sin(t-t_0) \\ -\sin(t-t_0) & \cos(t-t_0) \end{pmatrix}.$$

write

$$\varphi_1 : t \mapsto \exp(A(t-t_0))\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos(t-t_0) \\ -\sin(t-t_0) \end{pmatrix} \quad \text{and}$$

$$\varphi_2 : t \mapsto \exp(A(t-t_0))\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \sin(t-t_0) \\ \cos(t-t_0) \end{pmatrix}.$$

Then (φ_1, φ_2) is a basis of solutions by above considerations.

Let $(\underline{\varphi}_1, \dots, \underline{\varphi}_r)$ be a fundamental set of solutions of $\dot{\underline{x}} = A(t)\,\underline{x}$.

$\underline{\varphi}_j : I \longrightarrow \mathbb{R}^r$. Fix the standard basis on $\mathbb{R}^r$

$$\left(\underline{e}_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \dots, \underline{e}_r = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} \right)$$

Then can write $\underline{\varphi}_j$ in coordinates:

$$\underline{\varphi}_j : t \longmapsto \begin{pmatrix} \varphi_{1j}(t) \\ \vdots \\ \varphi_{rj}(t) \end{pmatrix}, \quad \text{where } \varphi_{ij} : I \longrightarrow \mathbb{R}$$

and Ψ_{t_0} is represented by the matrix

$$\left[\Psi_{t_0} \right]^{\text{standard basis of } \mathbb{R}^r}_{(\underline{\varphi}_1, \dots, \underline{\varphi}_r)} = \begin{pmatrix} \varphi_{11}(t_0) & \cdots & \varphi_{1r}(t_0) \\ \vdots & \ddots & \vdots \\ \varphi_{r1}(t_0) & \cdots & \varphi_{rr}(t_0) \end{pmatrix}.$$

This is often denoted by $\Psi(t_0)$ (without any reference to the basis!) and called <u>a fundamental matrix</u> of $(*)$.

<u>Example:</u> $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$ (**)

For any $t_0 \in I$, let

$$\varphi_1 : t \longmapsto \begin{pmatrix} \cos(t-t_0) \\ -\sin(t-t_0) \end{pmatrix}, \quad \varphi_2 : t \longmapsto \begin{pmatrix} \sin(t-t_0) \\ \cos(t-t_0) \end{pmatrix}$$

, as before

Then (φ_1, φ_2) is a basis for solutions of (**), as discussed above. For any $t_1 \in I$,

$$\Psi(t_1) = \left[\Psi_{t_1} \right]_{(\varphi_1, \varphi_2)}^{\text{standard basis of } \mathbb{R}^2} = \begin{pmatrix} \cos(t_1 - t_0) & \sin(t_1 - t_0) \\ -\sin(t_1 - t_0) & \cos(t_1 - t_0) \end{pmatrix}.$$

Frequently, $\Psi(t)$ is thought of as a function of t. In the example,

$$\Psi(t) = \begin{pmatrix} \cos(t-t_0) & \sin(t-t_0) \\ -\sin(t-t_0) & \cos(t-t_0) \end{pmatrix}$$

is a fundamental matrix of (*).

We'll return to constructions related to $\Psi(t)$ in the next lecture.

— // —

Def. Let $\varphi_1, \dots, \varphi_r : I \to \mathbb{R}^r$ be a collection of $\mathbb{R}^r$-valued functions. Fix the standard basis of $\mathbb{R}^r$ and write

$$\varphi_j : t \longmapsto \begin{pmatrix} \varphi_{1j}(t) \\ \vdots \\ \varphi_{rj}(t) \end{pmatrix}, \qquad \varphi_{ij} : I \to \mathbb{R}.$$

The <u>Wronskian</u> of $\varphi_1, \dots, \varphi_r$ is defined as the function

$$W(\varphi_1, \dots, \varphi_r) : t \mapsto \det \begin{pmatrix} \varphi_{11}(t) & \cdots & \varphi_{1r}(t) \\ \vdots & \ddots & \vdots \\ \varphi_{r1}(t) & \cdots & \varphi_{rr}(t) \end{pmatrix}.$$

<u>Example</u>: $\varphi_1 : t \mapsto \begin{pmatrix} \cos(t) \\ -\sin(t) \end{pmatrix}$, $\varphi_2 : t \longmapsto \begin{pmatrix} \sin(t) \\ \cos(t) \end{pmatrix}$.

$$W(\varphi_1, \varphi_2)(t) = \det \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix}$$

$$= \cos^2(t) + \sin^2(t) = 1.$$

<u>Theorem</u> Let $\varphi_1, \dots, \varphi_r : I \to \mathbb{R}^r$ be solutions of

$\dot{\underline{x}} = A(t)\underline{x}$, with $A(t)$ continuous over $I \subset \mathbb{R}$.

$(I \neq \emptyset)$

The following are equivalent:

i) $\varphi_1, \dots, \varphi_r$ are linearly dependent.

ii) $W(\varphi_1, \dots, \varphi_r)(t) = 0$ for all $t \in I$.

iii) $W(\varphi_1, \dots, \varphi_r)(t_0) = 0$ for a fixed $t_0 \in I$.

<u>Proof</u>: <u>i) implies ii)</u>
Suppose $\varphi_1, \dots, \varphi_r$ are linearly dependent.

Then there exist $a_1, \dots, a_r \in \mathbb{R}$ not all zero so that

$$a_1\varphi_1 + \dots + a_r\varphi_r = 0 \quad \text{(as functions } I \to \mathbb{R}^r\text{)}.$$

Then for each $t \in I$,

$$a_1\varphi_1(t) + \dots + a_r\varphi_r(t) = 0.$$

So, for each $t \in I$, the columns of the matrix

$$\begin{pmatrix} | & & | \\ \varphi_1(t) & \cdots & \varphi_r(t) \\ | & & | \end{pmatrix}$$

are linearly dependent. But then for each $t \in I$,

$$\det \begin{pmatrix} | & & | \\ \varphi_1(t) & \cdots & \varphi_r(t) \\ | & & | \end{pmatrix} = 0.$$

$$\parallel$$

$$W(\varphi_1, \dots, \varphi_r)(t)$$

ii) implies iii) This is clear as long as $I \neq \emptyset$.

iii) implies i)

Suppose that

$$W(\varphi_1, \ldots, \varphi_r)(t_0) = \det \begin{pmatrix} | & & | \\ \varphi_1(t_0) & \cdots & \varphi_r(t_0) \\ | & & | \end{pmatrix} = 0$$

for some $t_0 \in I$.

Then $\varphi_1(t_0), \ldots, \varphi_r(t_0)$ are linearly dependent.

So, there exist $a_1, \ldots, a_r \in \mathbb{R}$ not all zero so that

$$a_1 \varphi_1(t_0) + \cdots + a_r \varphi_r(t_0) = 0 \quad \left(\text{as vectors in } \mathbb{R}^r\right)$$

Define $\varphi = a_1 \varphi_1 + \cdots + a_r \varphi_r$.

By linearity, φ is a solution of $\dot{\underline{x}} = A(t)\underline{x}$ (since $\varphi_1, \ldots, \varphi_r$ are solutions).

Because $\varphi(t_0) = a_1 \varphi_1(t_0) + \cdots + a_r \varphi_r(t_0) = 0$, it follows by the uniqueness theorem that $\varphi = 0$ (the zero solution).

But this means that there exist $a_1, \dots, a_r$ not all zero

so that $a_1 \varphi_1 + \cdots + a_r \varphi_r = 0$.

Therefore, $\varphi_1, \dots, \varphi_r$ are linearly dependent. $\square$

<u>Corollary.</u> Let $\varphi_1, \dots, \varphi_r$ be solutions of $\dot{\underline{x}} = A(t)\underline{x}$.

The following are equivalent:

i) $\varphi_1, \dots, \varphi_r$ are linearly independent.

ii) $W(\varphi_1, \dots, \varphi_r)(t_0) \neq 0$ for a $t_0 \in I$

iii) $W(\varphi_1, \dots, \varphi_r)(t) \neq 0$ for all $t \in I$.

<u>Remark.</u> Consider the equation

$$(L) \qquad \frac{d^r y}{dt^r} + a_{r-1}(t)\frac{d^{r-1} y}{dt^{r-1}} + \cdots + a_1(t)\frac{dy}{dt} + a_0(t) y = 0$$

The system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ &\vdots \\ \dot{x}_r &= -a_{r-1}(t)x_r - \cdots - a_1(t)x_2 - a_0(t)x_1 \end{aligned}$$

is equivalent to the equation (L), in the sense that we have the following vector space isomorphism:

$$y \longmapsto (y, \dot{y}, \ddot{y}, \dots, y^{(r-1)})$$

$$\left\{ \begin{array}{c} \text{Solutions of} \\ \dfrac{d^r y}{dt^r} + \dots + a_0(t) y = 0 \end{array} \right\} \cong \left\{ \begin{array}{c} \text{Solutions of} \\ \dot{x}_1 = x_2 \\ \vdots \\ \dot{x}_r = -a_{r-1}(t) x_r - \dots - a_0(t) x_1 \end{array} \right\}$$

$$x_1 \longleftarrow (x_1, x_2, \dots, x_r)$$

Suppose that $y_1, \dots, y_r : I \to \mathbb{R}$ is a collection of solutions of (L).

Let $\varphi_j = (y_j, \dot{y}_j, \ddot{y}_j, \dots, y_j^{(r-1)})$, $j = 1, \dots, r$

be the corresponding solutions of the equivalent first-order system.

Then

$$W(\varphi_1, \dots, \varphi_r)(t) = \det \begin{pmatrix} y_1 & \cdots & y_r \\ \dot{y}_1 & \cdots & \dot{y}_r \\ \vdots & \ddots & \vdots \\ y_1^{(r-1)} & \cdots & y_r^{(r-1)} \end{pmatrix}.$$

This recovers the previous definition of the Wronskian!