Mthe 237

Lecture 33

Nov. 24, 2017

Topic: Another look at second-order linear (homogeneous) equations

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Now that we have developed a way to solve systems

$$\dot{\underline{x}} = A\underline{x},$$

where A is a constant matrix, and have a qualitative understanding of the behaviour of the solutions, let's take another look at the equation

$$\frac{d^2y}{dt^2} + 2\gamma\frac{dy}{dt} + \omega_0^2 y = 0, \qquad \begin{pmatrix} \gamma \geq 0 \\ \omega_0 \neq 0 \end{pmatrix}$$

which can be studied through its equivalent first-order system

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -2\gamma x_2 - \omega_0^2 x_1$$

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ -\omega_0^2 & -2\gamma \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

By a problem in Homework 9, we have

$$\det(A - zI) = p_A(z) = \chi(z) = z^2 + 2\gamma z + \omega_0^2.$$

Since the computation is short, let's verify:

$$\det(A - zI) = \det\left(\begin{pmatrix} -z & 1 \\ -\omega_0^2 & -2\gamma - z \end{pmatrix}\right)$$

$$= (-z)(-2\gamma - z) + \omega_0^2$$

$$= z^2 + 2\gamma z + \omega_0^2, \quad \text{as expected.}$$

The kinds of roots $p_A(z)$ has depends on the discriminant $\Delta = "b^2 - 4ac" = 4\gamma^2 - 4\omega_0^2$

$$= 4(\gamma^2 - \omega_0^2).$$

By the quadratic formula, the roots are

$$\frac{-2\gamma \pm \sqrt{\Delta}}{2}$$

So we see that:

- If $\Delta > 0$, there are two distinct real roots
- If $\Delta = 0$, there is a single double root
- If $\Delta < 0$, there are two conjugate complex roots

Suppose $\Delta > 0$. Two distinct real eigenvalues.

(We called this the overdamped case before.)

The eigenvalues are

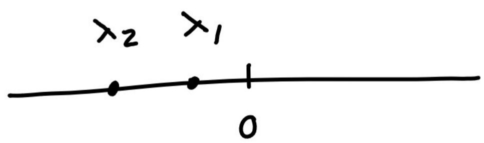
$$\lambda_1 = -\gamma + \sqrt{\gamma^2 - \omega_0^2} \quad \text{and}$$

$$\lambda_2 = -\gamma - \sqrt{\gamma^2 - \omega_0^2}$$

Since $\omega_0^2 > 0$, $\gamma^2 - \omega_0^2 < \gamma^2$, so $\sqrt{\gamma^2 - \omega_0^2} < \gamma$.

(In the start, we assumed $\gamma \geq 0$ and $\omega_0 \neq 0$)

Therefore, both eigenvalues are negative.



Finding eigenvectors:

$$\ker(A - \lambda_1 I) = \ker \begin{pmatrix} \gamma - \sqrt{\gamma^2 - \omega_0^2} & 1 \\ -\omega_0^2 & -\gamma - \sqrt{\gamma^2 - \omega_0^2} \end{pmatrix}$$

$$\underline{v}_1 = \begin{pmatrix} 1 \\ -(\gamma - \sqrt{\gamma^2 - \omega_0^2}) \end{pmatrix} = \begin{pmatrix} 1 \\ -\gamma + \sqrt{\gamma^2 - \omega_0^2} \end{pmatrix}$$

is an eigenvector corresponding to λ_1.

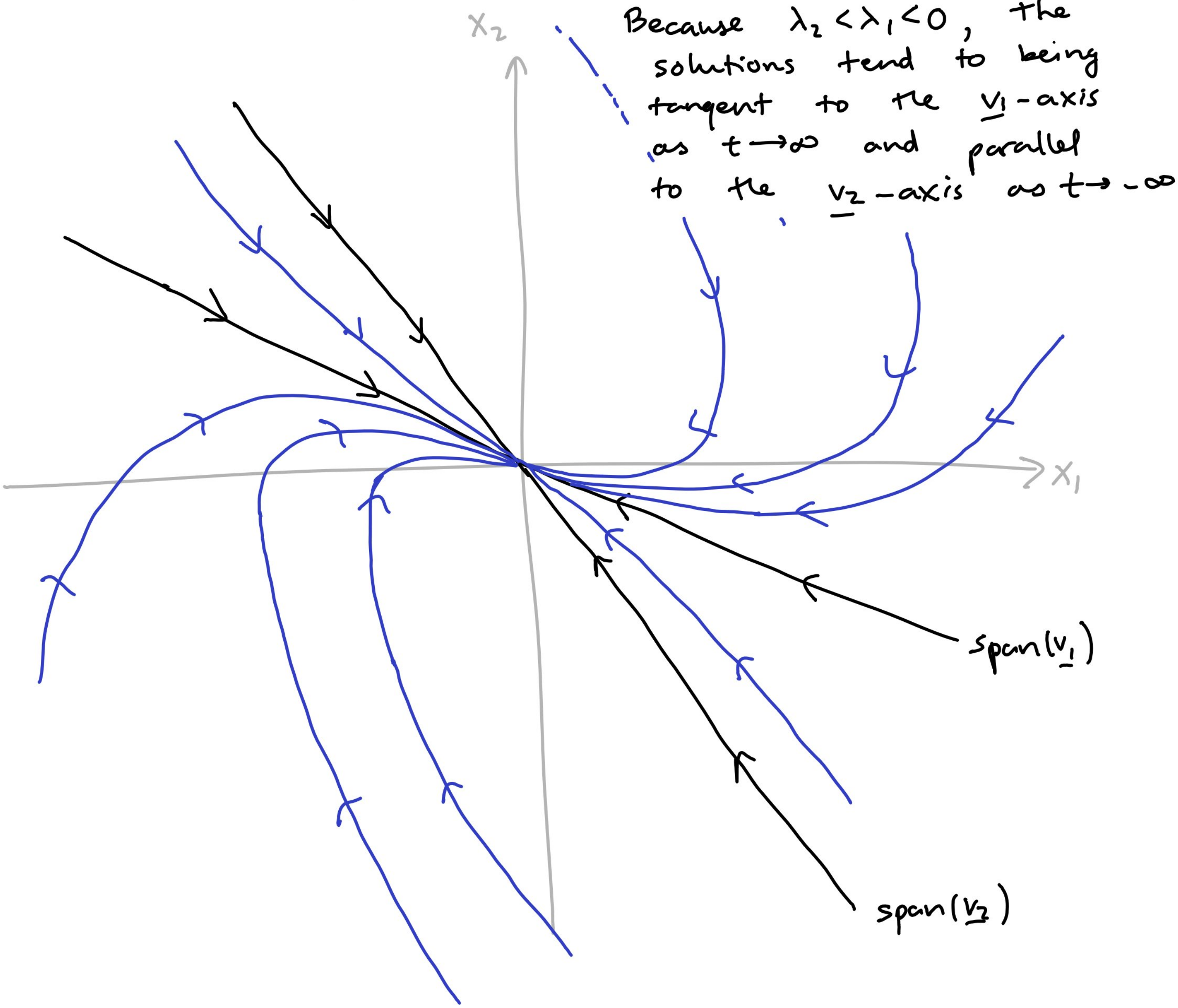
Similarly,

$$\ker\left(A-\lambda_2 I\right) = \ker\begin{pmatrix} \gamma+\sqrt{\gamma^2-\omega_0^2} & 1 \\ -\omega_0^2 & -\gamma+\sqrt{\gamma^2-\omega_0^2} \end{pmatrix}$$

$$\underline{v_2} = \begin{pmatrix} 1 \\ -\gamma-\sqrt{\gamma^2-\omega_0^2} \end{pmatrix}$$ is an eigenvector corresponding to λ_2.

The solution portrait is a stable node:

Because $\lambda_2 < \lambda_1 < 0$, the solutions tend to being tangent to the $\underline{v_1}$-axis as $t \to \infty$ and parallel to the $\underline{v_2}$-axis as $t \to -\infty$

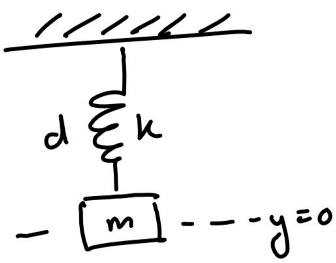


As we have seen, the equation

$$\frac{d^2y}{dt^2} + 2\gamma \frac{dy}{dt} + \omega_0^2 y = 0 \qquad (*)$$

is the equation of motion of a damped harmonic oscillator.

The running example throughout the course has been $y =$ displacement of a mass hanging from a damped spring



$$\omega_0 = \sqrt{\frac{k}{m}}, \qquad \gamma = \frac{d}{2m}.$$

The correspondence between solutions of equation $(*)$ and its equivalent first-order system is given by

$$y \longmapsto (y, \dot{y})$$

$$\left\{ \begin{array}{l} \text{Solutions of} \\ \dfrac{d^2y}{dt^2} + 2\gamma\dfrac{dy}{dt} + \omega_0^2 y = 0 \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{Solutions of} \\ \dot{x}_1 = x_2 \\ \dot{x}_2 = -2\gamma x_2 - \omega_0^2 x_1 \end{array} \right\}$$

$$x_1 \longleftarrow\!\shortmid (x_1, x_2)$$

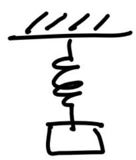
Therefore, in the case when $(*)$ models the motion of a damped harmonic oscillator, the functions x_1 and x_2 can be thought of as the functions y and $\dot{y}$.

In the case of the mass on a spring, we can think of x_1 and x_2 as

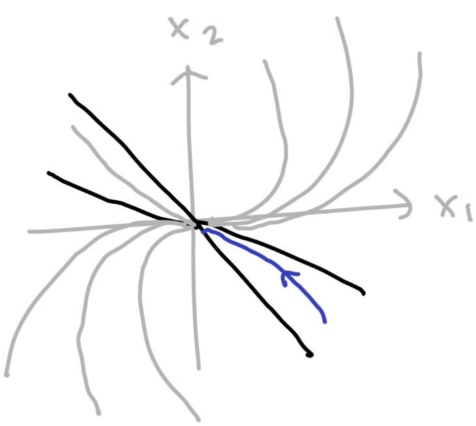
x_1 — displacement of mass from rest position

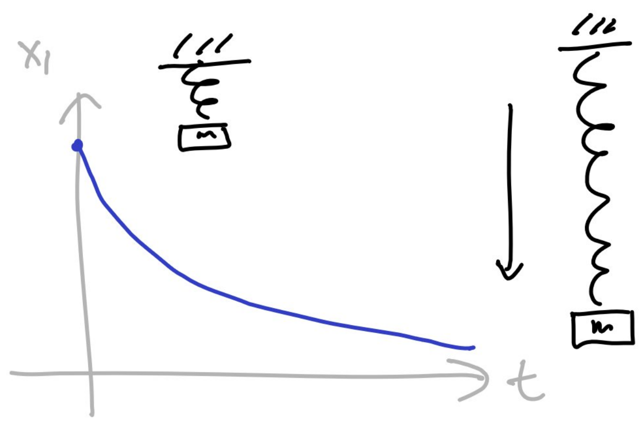
x_2 — velocity of mass.

To obtain information about the position of the mass, it is enough to only look at the x_1-coordinate of the solution / flow line (this corresponds to looking only at the projection or "shadow" of the flow line onto the x_1-axis.)

Let's look at three solutions / flow lines and the motions of  they correspond to:

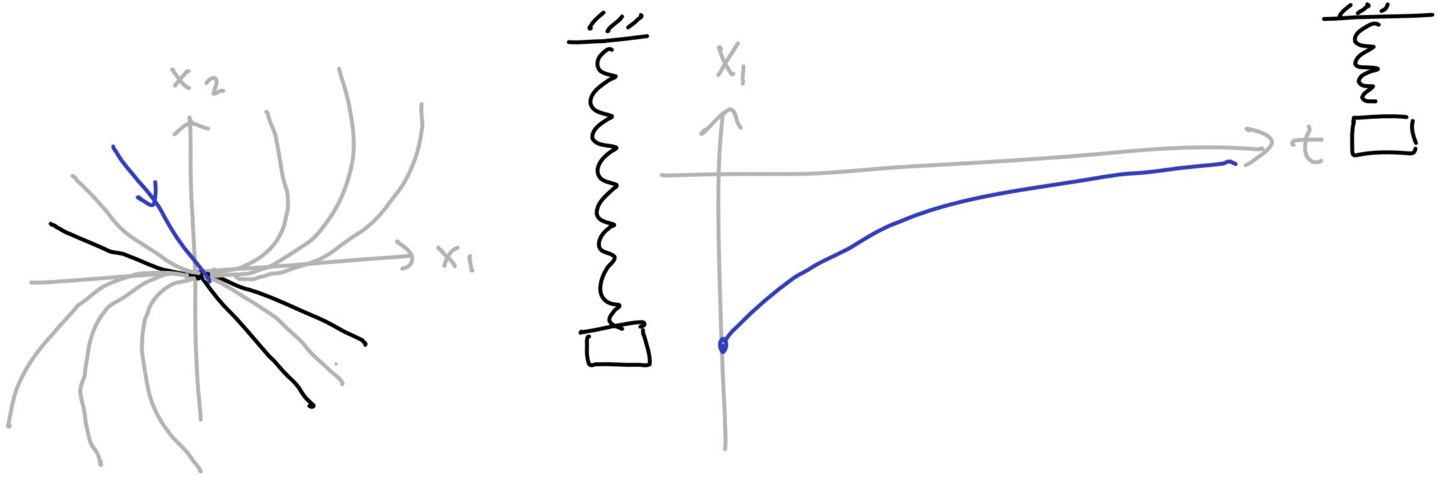
1)





(This graph is obtained from the blue flow line by keeping track of only the x_1 coordinate as a function of time.)

2)

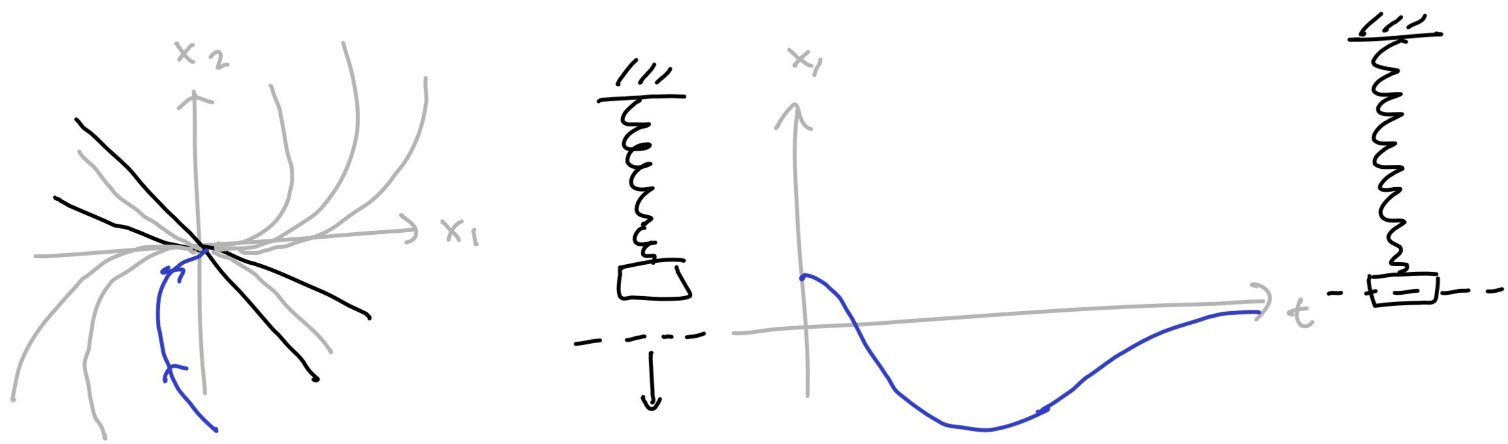


Although the flow is a line, the flow line picture does not keep track of the <u>speed</u> of the motion of the flow. We know from the matrix exponential that the solution is

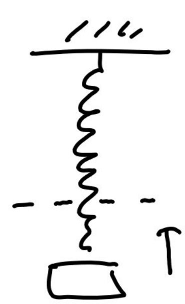
$$\underline{x}(t) = a e^{\lambda_2 t} \underline{v_2}.$$

Since $\lambda_2 < 0$, $e^{\lambda_2 t} \to 0$ as $t \to \infty$.

3)



This kind of flow line corresponds to the other characteristic motion of an overdamped harmonic oscillator (passing through its rest point once).



Carrying out the computation of the matrix exponential,

$$\exp\left(\begin{pmatrix} 0 & 1 \\ -\omega_0^2 & -2\gamma \end{pmatrix} t\right) = Q \begin{pmatrix} e^{(-\gamma+\sqrt{\gamma^2-\omega_0^2})t} & 0 \\ 0 & e^{(-\gamma-\sqrt{\gamma^2-\omega_0^2})t} \end{pmatrix} Q^{-1}$$

where $Q = \begin{pmatrix} 1 & 1 \\ -\gamma+\sqrt{\gamma^2-\omega_0^2} & -\gamma-\sqrt{\gamma^2-\omega_0^2} \end{pmatrix}$.

writing

$$\lambda_1 = -\gamma + \sqrt{\gamma^2-\omega_0^2}$$
$$\lambda_2 = -\gamma - \sqrt{\gamma^2-\omega_0^2},$$

$$\exp\left(\begin{pmatrix} 0 & 1 \\ -\omega_0^2 & -2\gamma \end{pmatrix} t\right) = \frac{1}{\lambda_2-\lambda_1} \begin{pmatrix} \lambda_2 e^{\lambda_1 t} - \lambda_1 e^{\lambda_2 t} & e^{\lambda_2 t} - e^{\lambda_1 t} \\ \lambda_1\lambda_2\left(e^{\lambda_1 t} - e^{\lambda_2 t}\right) & \lambda_2 e^{\lambda_2 t} - \lambda_1 e^{\lambda_1 t} \end{pmatrix}$$

In particular, with initial conditions $\begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} x_0 \\ v_0 \end{pmatrix}$,

$$x_1(t) = \left(\frac{\lambda_2 e^{\lambda_1 t} - \lambda_1 e^{\lambda_2 t}}{\lambda_2-\lambda_1}\right) x_0 + \left(\frac{e^{\lambda_2 t} - e^{\lambda_1 t}}{\lambda_2-\lambda_1}\right) v_0$$

$$= \left(\frac{\lambda_2 x_0 - v_0}{\lambda_2-\lambda_1}\right) e^{\lambda_1 t} + \left(\frac{v_0 - \lambda_1 x_0}{\lambda_2-\lambda_1}\right) e^{\lambda_2 t}.$$

We know from before that a basis of solutions of

$$\frac{d^2y}{dt^2} + 2\gamma \frac{dy}{dt} + \omega_0^2 y = 0 \qquad (*)$$

when $\gamma^2 - \omega_0^2 > 0$ is given by $\{e^{\lambda_1 t}, e^{\lambda_2 t}\}$.

What is the solution of $(*)$ with initial conditions

$$y(0) = x_0$$
$$\frac{dy}{dt}(0) = v_0 \; ?$$

writing $y(t) = Ae^{\lambda_1 t} + Be^{\lambda_2 t}$, we see that we should have

$$y(0) = A + B = x_0$$
$$\frac{dy}{dt}(0) = \lambda_1 A + \lambda_2 B = v_0 .$$

$$\begin{pmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} x_0 \\ v_0 \end{pmatrix} \Rightarrow \begin{pmatrix} A \\ B \end{pmatrix} = \frac{1}{\lambda_2 - \lambda_1}\begin{pmatrix} \lambda_2 & -1 \\ -\lambda_1 & 1 \end{pmatrix}\begin{pmatrix} x_0 \\ v_0 \end{pmatrix}$$

$$A = \left(\frac{\lambda_2 x_0 - v_0}{\lambda_2 - \lambda_1}\right) \quad \text{and} \quad B = \left(\frac{v_0 - \lambda_1 x_0}{\lambda_2 - \lambda_1}\right),$$

in agreement with what we just obtained using the matrix exponential! :)

Another case: Simple Harmonic Motion.

$$\frac{d^2y}{dt^2} + \omega_0^2 y = 0 \,, \quad \omega_0 \neq 0 \qquad \left(\gamma = 0 \text{ in } (*) \right)$$

The associated first-order system is

$$\dot{x}_1 = x_2 \qquad \qquad \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\omega_0^2 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$
$$\dot{x}_2 = -\omega_0^2 x_1$$

The characteristic polynomial is

$$\det \begin{pmatrix} -z & 1 \\ -\omega_0^2 & -z \end{pmatrix} = z^2 + \omega_0^2$$
$$= (z + i\omega_0)(z - i\omega_0)$$

There are two purely imaginary eigenvalues

$$\lambda_1 = i\omega_0$$
$$\lambda_2 = \overline{\lambda_1} = -i\omega_0.$$

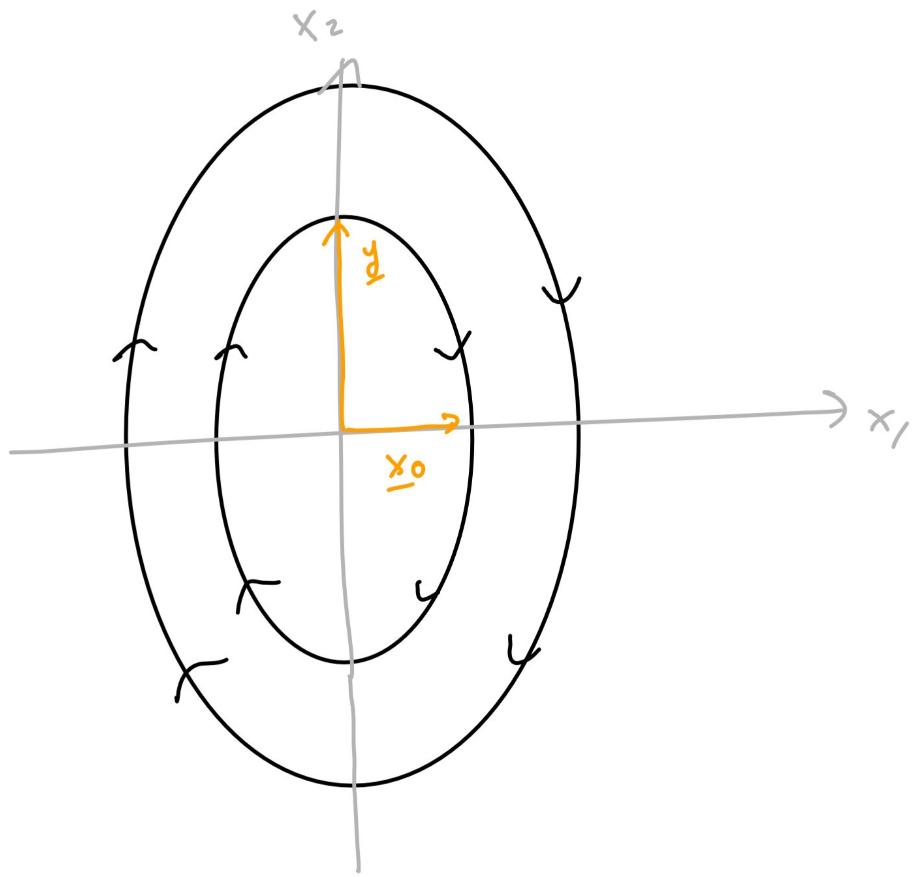
Finding a complex eigenvector –

$$\ker \begin{pmatrix} -i\omega_0 & 1 \\ -\omega_0^2 & -i\omega_0 \end{pmatrix}$$

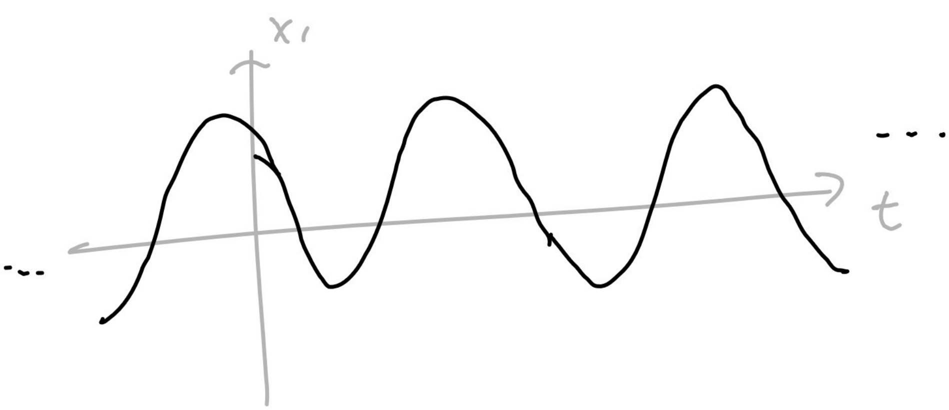
$$\underline{v_1} = \begin{pmatrix} 1 \\ i\omega_0 \end{pmatrix} \qquad \left(\text{Corresponds to eigenvalue } \lambda_1 = i\omega_0 \right)$$

Taking the real and imaginary parts of $\underline{v}_1$,

$$\underline{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \underline{y} = \begin{pmatrix} 0 \\ w_0 \end{pmatrix}$$



Flow lines are ovals. Keeping track of only the x_1-coordinate, we see sinusoidal oscillation:

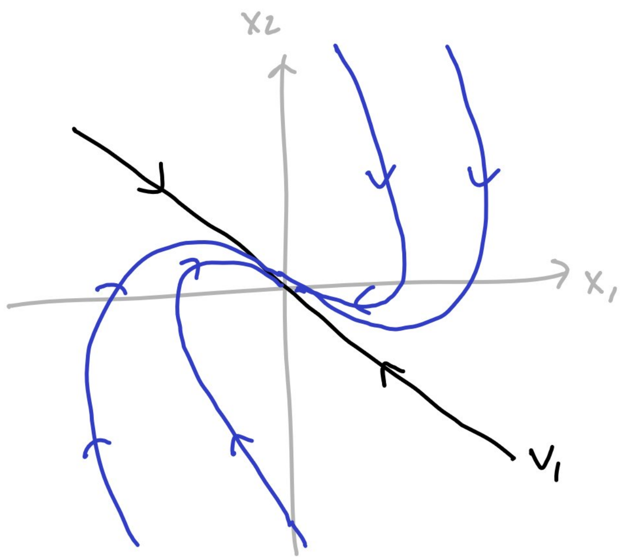


This is exactly what we expect the motion of a simple harmonic oscillator to be.

Due to lack of time, I was not able to talk about the flow lines corresponding to underdamped and critically damped cases.

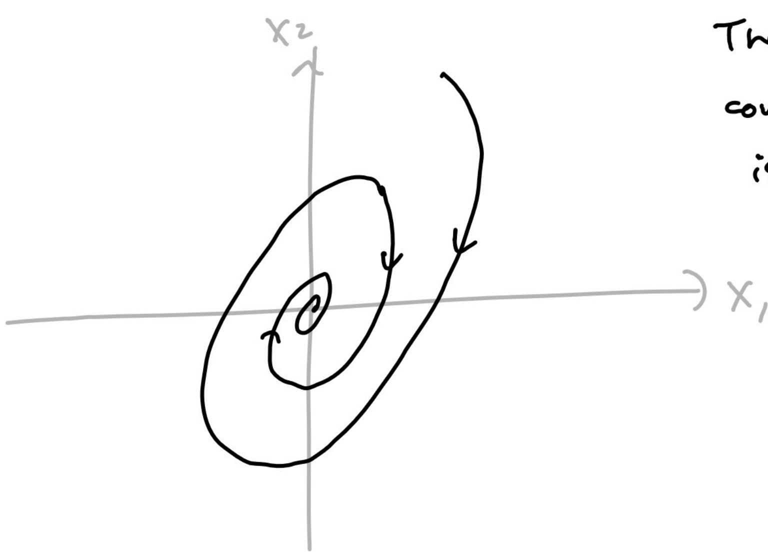
It is a challenging exercise to work out the details.

<u>Critically damped</u>: $\Delta = 0$. Single eigenvalue of (algebraic) multiplicity 2.



Turns out that the first-order system is not diagonalizable.

<u>Underdamped</u>: $\Delta < 0$. Two complex eigenvalues.



The real part of the complex eigenvalues is $-\gamma \leq 0$.

The flow lines are centres or stable spirals.