Mthe 237
Lecture 32
Nov. 22, 2017

Topic: Qualitative picture of flow lines for two-variable first-order systems

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Theorem The solution of the first-order system

$$\dot{\underline{x}} = A\,\underline{x}\,, \quad \underline{x}(t_0) = \underline{x}_0 \qquad (*)$$

is given by

$$\underline{x}(t) = \exp\left(A(t-t_0)\right)\underline{x}_0\,, \quad t \in \mathbb{R}$$

Proof sketch: Writing out the defining series on the right side,

$$\exp(A(t-t_0))\,\underline{x}_0 = \left(I + A(t-t_0) + \frac{A^2(t-t_0)^2}{2!} + \cdots \right)\underline{x}_0$$

Differentiating term-by-term with respect to t,

This is the part of the proof that requires justification. It is not always true that differentiating a series term-by-term gives a series that converges to the derivative of the function defined by the initial series. However, it is true for series with a certain type of convergence, and the matrix exponential series does have this convergence property.

$$\left(\dot{\exp(A(t-t_0))\,\underline{x}_0}\right) = \left(0 + A\cdot 1 + \frac{A^2\cdot 2(t-t_0)}{2!} + \frac{A^3\cdot 3(t-t_0)^2}{3!} + \cdots\right)\underline{x}_0$$

$$= A\left(I + A(t-t_0) + \frac{A^2(t-t_0)^2}{2!} + \cdots\right)\underline{x}_0$$

$$= A\exp(A(t-t_0))\,\underline{x}_0.$$

So $\underline{x}(t) = \exp(A(t-t_0))\,\underline{x}_0$ satisfies $\dot{\underline{x}} = A\underline{x}$.

Moreover, $\exp(A(t-t_0))\Big|_{t=t_0} = \exp(\text{zero matrix})$

$$= I,$$

Can be seen from the defining series
$\exp(0) = I + 0 + \frac{0^2}{2!} + \cdots$

So $\underline{x}(t_0) = I\,\underline{x}_0 = \underline{x}_0.$

$\underline{x}(t) = \exp(A(t-t_0))\,\underline{x}_0$ also satisfies the initial condition.

By the uniqueness theorem for linear systems (A is certainly continuous with t, being constant), this is <u>the</u> solution of (*).

§ Computing $\exp(A(t-t_0))$

If $A = Q \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_r \end{pmatrix} Q^{-1}$,

$$A(t-t_0) = Q \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_r \end{pmatrix} Q^{-1} (t-t_0)$$

$$= Q \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_r \end{pmatrix} (t-t_0)\, Q^{-1} \quad \Bigg\{ \text{multiplication by a constant commutes with every matrix}$$

$$= Q \begin{pmatrix} \lambda_1(t-t_0) & & \\ & \ddots & \\ & & \lambda_r(t-t_0) \end{pmatrix} Q^{-1}$$

So

$$\exp(A(t-t_0)) = Q \begin{pmatrix} e^{\lambda_1(t-t_0)} & & \\ & \ddots & \\ & & e^{\lambda_r(t-t_0)} \end{pmatrix} Q^{-1}$$

If $A = Q \begin{pmatrix} \sigma & \omega \\ -\omega & \sigma \end{pmatrix} Q^{-1}$

$$A(t-t_0) = Q \begin{pmatrix} \sigma(t-t_0) & \omega(t-t_0) \\ -\omega(t-t_0) & \sigma(t-t_0) \end{pmatrix} Q^{-1}$$

$$\exp(A(t-t_0)) = Q \begin{pmatrix} e^{\sigma(t-t_0)}\cos(\omega(t-t_0)) & e^{\sigma(t-t_0)}\sin(\omega(t-t_0)) \\ -e^{\sigma(t-t_0)}\sin(\omega(t-t_0)) & e^{\sigma(t-t_0)}\cos(\omega(t-t_0)) \end{pmatrix} Q^{-1}$$

If $A = Q \begin{pmatrix} \lambda & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{pmatrix} Q^{-1}$, ($s \times s$ real Jordan block)

$$A(t-t_0) = Q \begin{pmatrix} \lambda(t-t_0) & t-t_0 & & \\ & \ddots & \ddots & \\ & & \ddots & t-t_0 \\ & & & \lambda(t-t_0) \end{pmatrix} Q^{-1}$$

and

$$\exp\left(A(t-t_0)\right) = Q\, e^{\lambda(t-t_0)} \begin{pmatrix} 1 & (t-t_0) & \frac{(t-t_0)^2}{2!} & \cdots & \frac{(t-t_0)^{s-2}}{(s-2)!} & \frac{(t-t_0)^{s-1}}{(s-1)!} \\ 0 & 1 & (t-t_0) & \cdots & \frac{(t-t_0)^{s-3}}{(s-3)!} & \frac{(t-t_0)^{s-2}}{(s-2)!} \\ 0 & 0 & 1 & \cdots & \frac{(t-t_0)^{s-4}}{(s-4)!} & \frac{(t-t_0)^{s-3}}{(s-3)!} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & (t-t_0) \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix} Q^{-1}$$

(and a complex eigenvalue Jordan block is similar.)

Example.
$$\dot{x} = y, \qquad \dot{y} = -x, \qquad x(0) = x_0, \quad y(0) = y_0.$$

$$\dot{\underline{x}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \underline{x}_0.$$ A is already in the form $\begin{pmatrix} \sigma & \omega \\ -\omega & \sigma \end{pmatrix}$.

$$\underline{x}(t) = \exp(At)\,\underline{x}_0$$
$$= \begin{pmatrix} \cos(\omega t) & \sin(\omega t) \\ -\sin(\omega t) & \cos(\omega t) \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$
$$= \begin{pmatrix} \cos(\omega t) \\ -\sin(\omega t) \end{pmatrix} x_0 + \begin{pmatrix} \sin(\omega t) \\ \cos(\omega t) \end{pmatrix} y_0$$

$$A \sin(\omega t + \varphi) = A\cos(\omega t)\sin(\varphi) + A\sin(\omega t)\cos(\varphi)$$
$$A \cos(\omega t + \varphi) = A\cos(\omega t)\cos(\varphi) - A\sin(\omega t)\sin(\varphi)$$

Choose A and φ so that
$$x_0 = A\sin(\varphi) \qquad \left(A = \sqrt{x_0^2 + y_0^2} \right.$$
$$y_0 = A\cos(\varphi) \qquad \left. \frac{x_0}{y_0} = \tan\varphi \right)$$

Then
$$x_0 \cos(\omega t) + y_0 \sin(\omega t) = A \sin(\omega t + \varphi)$$
$$-x_0 \sin(\omega t) + y_0 \cos(\omega t) = A \cos(\omega t + \varphi)$$

So the solution may be written as

$$t \longmapsto \begin{pmatrix} A \sin(\omega t + \varphi) \\ A \cos(\omega t + \varphi) \end{pmatrix}, \qquad t \in \mathbb{R}$$

These are clockwise circles (as opposed to the more standard parametrization with cos in the first coordinate, which moves counterclockwise.)

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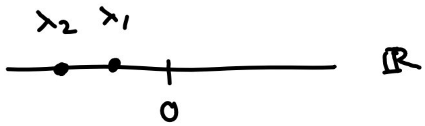
§ Flow lines of two-variable first-order systems

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}, \qquad \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

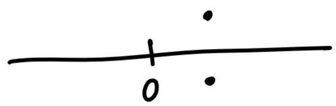
Let's understand the qualitative behaviour of the solutions of this system, as an application of our knowledge of the matrix exponential.

The types of behaviour will depend on the types (real/complex) of the eigenvalues of A, as well as the sign of the real part of the eigenvalue.

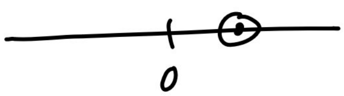
It is convenient to represent the eigenvalues diagramatically:



represents two distinct real eigenvalues with $\lambda_2 < \lambda_1 < 0$.



represents a pair of conjugate complex eigenvalues $\lambda, \bar{\lambda}$ with positive real part



represents a real eigenvalue of (algebraic) multiplicity two.

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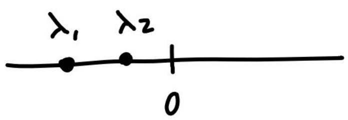


Let $\underline{v}_1$ and $\underline{v}_2$ be the corresponding eigenvectors.

In basis $\{\underline{v}_1, \underline{v}_2\}$, A takes the form

$$\begin{pmatrix} e^{\lambda_1} & 0 \\ 0 & e^{\lambda_2} \end{pmatrix}.$$

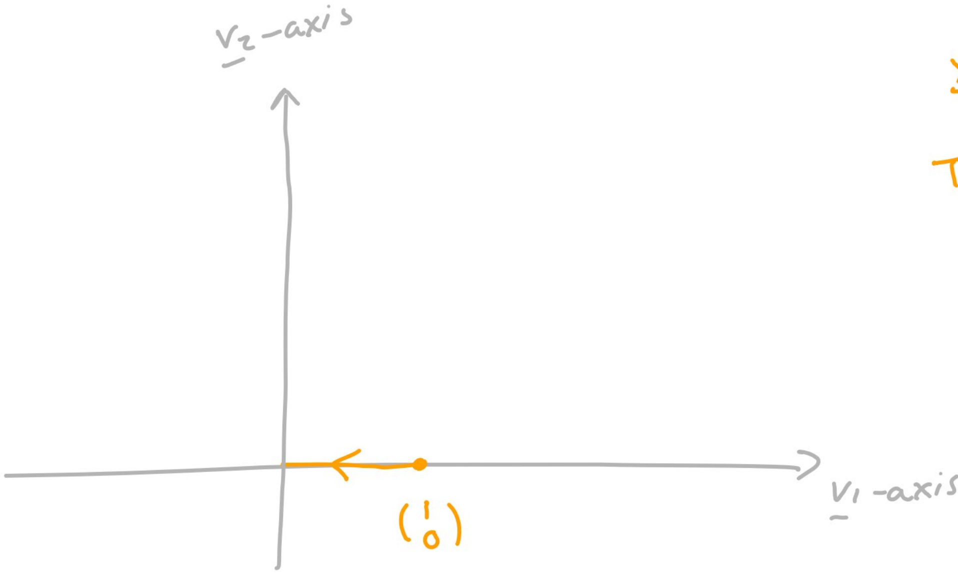
Suppose that $\underline{x}_0 = a\underline{v}_1 + b\underline{v}_2$.

So that $\underline{x}(t) = \exp(At)\,\underline{x}_0$

$$= \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a e^{\lambda_1 t} \\ b e^{\lambda_2 t} \end{pmatrix}$$

$$= a e^{\lambda_1 t}\,\underline{v}_1 + b e^{\lambda_2 t}\,\underline{v}_2.$$

Let's now consider the behaviour of the solutions for some representative types of the initial conditions.

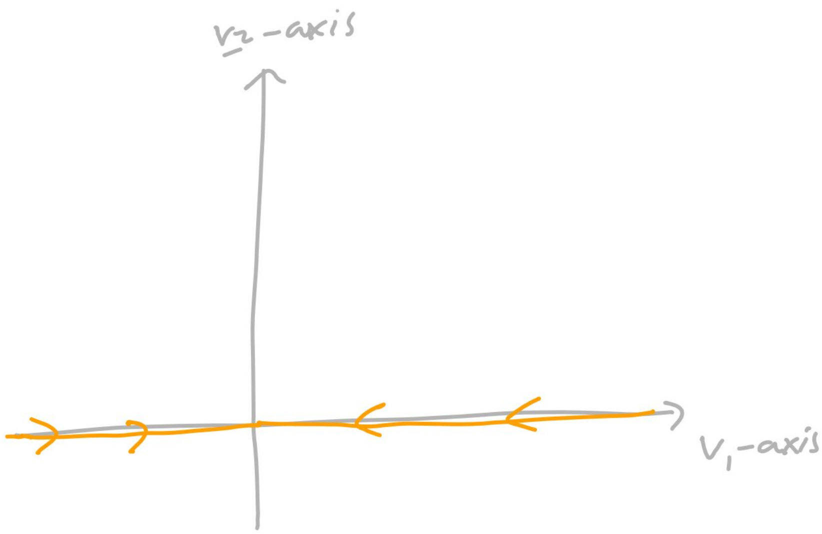


If $\begin{pmatrix}a\\b\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}$,

$$\underline{x}(t) = e^{\lambda_1 t}\,\underline{v}_1$$

This stays on the line spanned by $\underline{v}_1$.

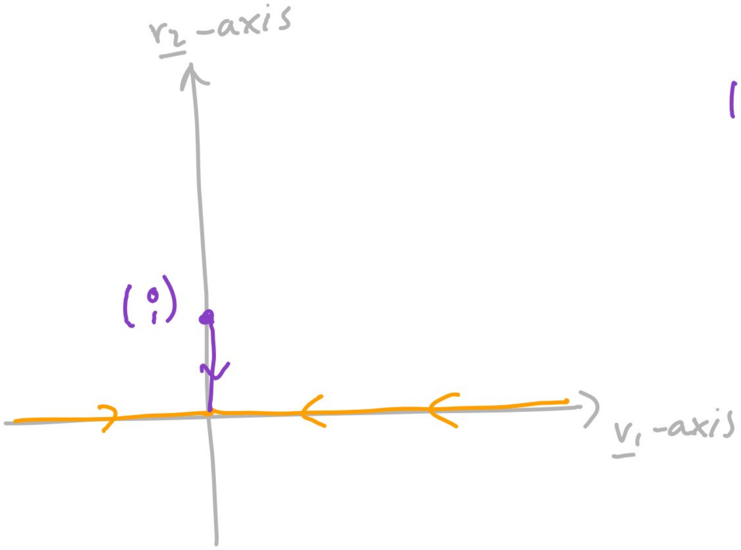
As $t \to \infty$, $\underline{x}(t) \to 0$



Similarly, if $\begin{pmatrix}a\\b\end{pmatrix} = \begin{pmatrix}a\\0\end{pmatrix}$

$$\underline{x}(t) = a\,e^{\lambda_1 t}\,\underline{v}_1.$$

This stays on the line span($\underline{v}_1$) and approaches the origin.

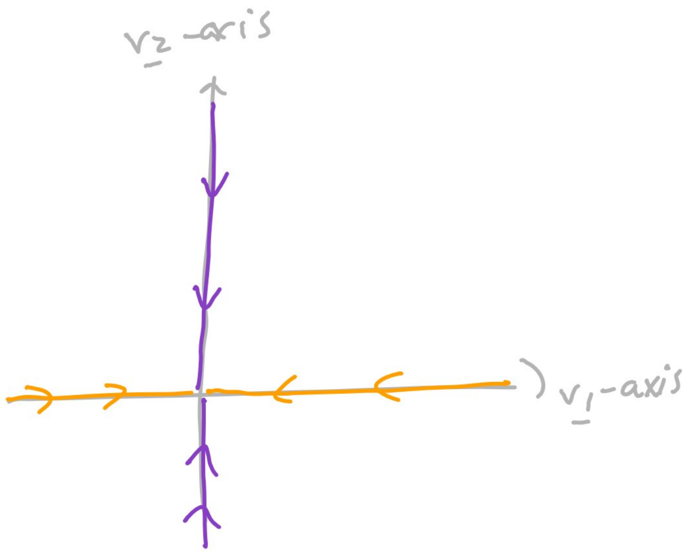


If $\begin{pmatrix}a\\b\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix}$, then

$$\underline{x}(t) = e^{\lambda_2 t}\,\underline{v}_2.$$

As $t \to \infty$, $\underline{x}(t) \to \underline{0}$.

Because both eigenvalues are real and negative, the behavior along the two eigenspaces is similar.



If $\begin{pmatrix}a\\b\end{pmatrix} = \begin{pmatrix}0\\b\end{pmatrix}$, see a similar flow.

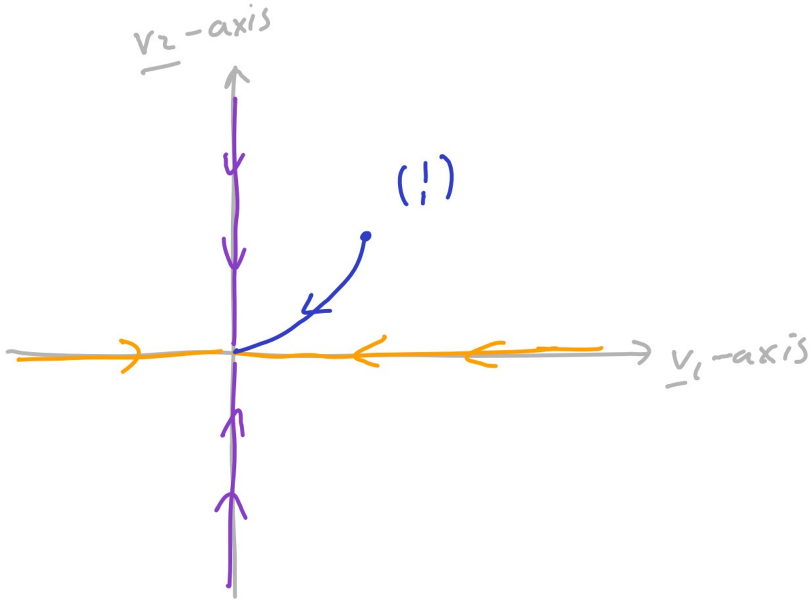
Now, starting at $\begin{pmatrix}a\\b\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix}$, for instance,

$$\underline{x}(t) = e^{\lambda_1 t}\,\underline{v}_1 + e^{\lambda_2 t}\,\underline{v}_2.$$

As $t \to \infty$, $\underline{x} \to \underline{0}$.

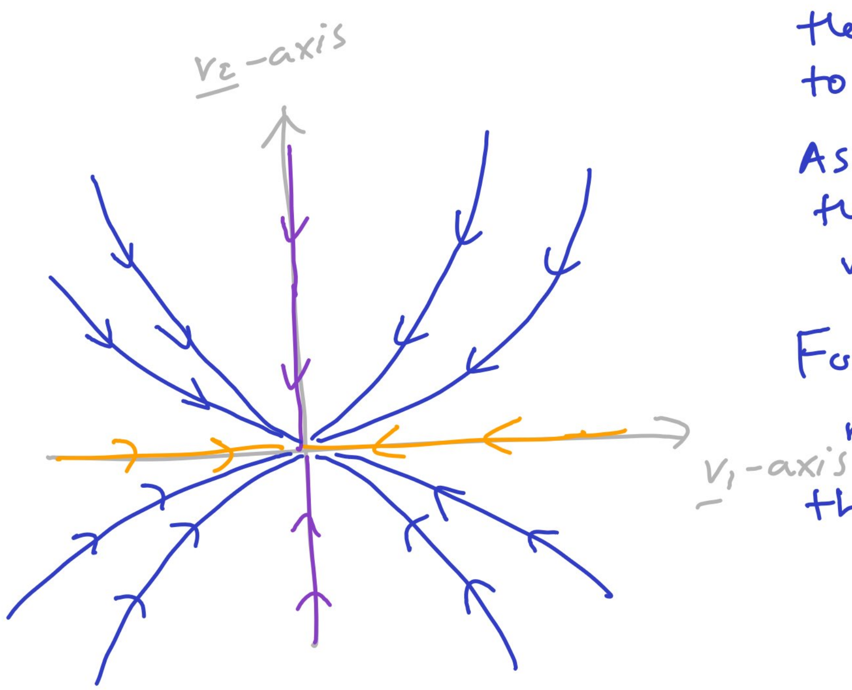
However, because $\lambda_1 \neq \lambda_2$, the convergence to 0 along the $\underline{v}_1$ and $\underline{v}_2$-axes occurs at <u>different rates</u>.

Because $e^{\lambda_2 t}$ goes to 0 faster than $e^{\lambda_1 t}$ ($\lambda_2 < \lambda_1 < 0$), the flow line will be curved.



Near the origin, the rates are nearly equal ($e^{\lambda_1 t} \sim e^{\lambda_2 t}$ for large t), so the flow is close to linear.

One can show analytically (as on the homework) that the flow lines are tangent to the $\underline{v}_1$-axis near the origin. As $t \to -\infty$, the direction of the flow lines approaches the vertical (by the same principle).



For other types of mixed initial conditions, the picture is similar.

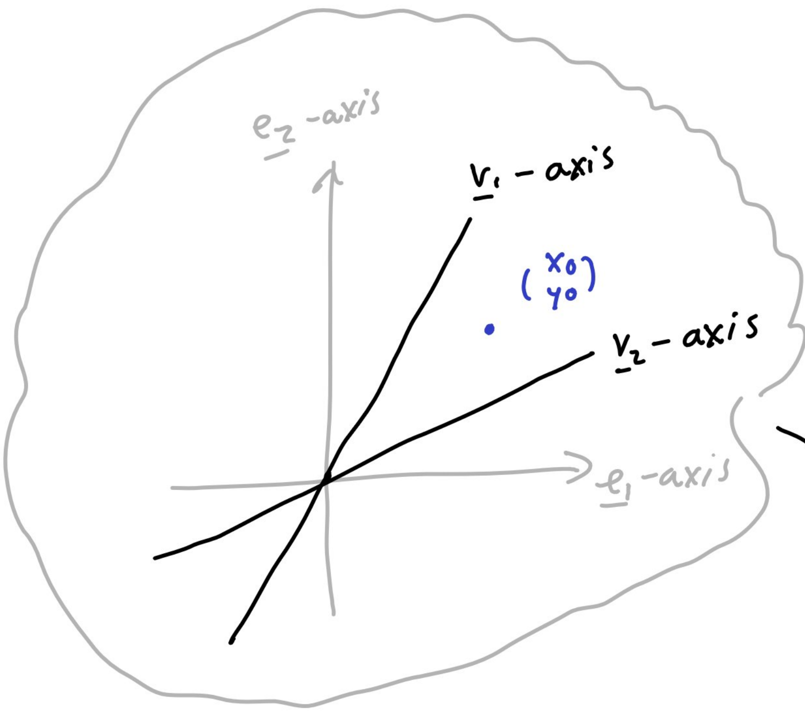
As a consequence of the uniqueness theorem, flow lines / solutions cannot cross, which helps with the sketch.

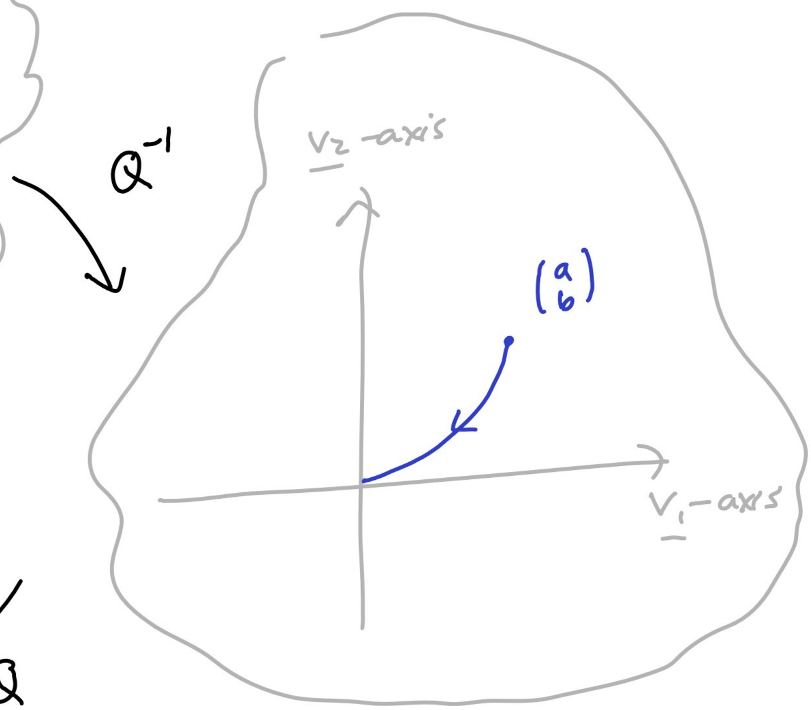
Finally, suppose $\underline{v}_1 = \begin{pmatrix} v_{11} \\ v_{12} \end{pmatrix}$ and $\underline{v}_2 = \begin{pmatrix} v_{21} \\ v_{22} \end{pmatrix}$ in the basis $\{\underline{e}_1, \underline{e}_2\}$ (for example, this could be the standard basis $\underline{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\underline{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$)

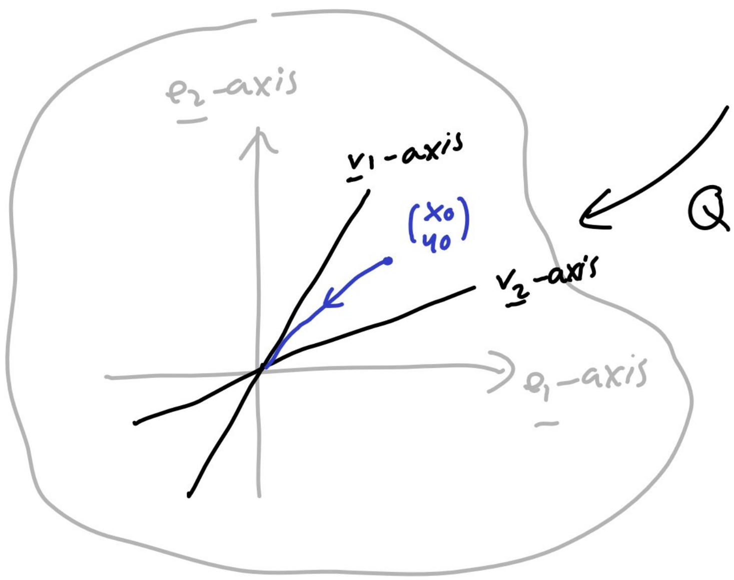
Then the solution satisfying $\underline{x}(0) = x_0 \underline{e}_1 + y_0 \underline{e}_2$ is

$$\underline{x}(t) = Q \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix} Q^{-1} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}.$$

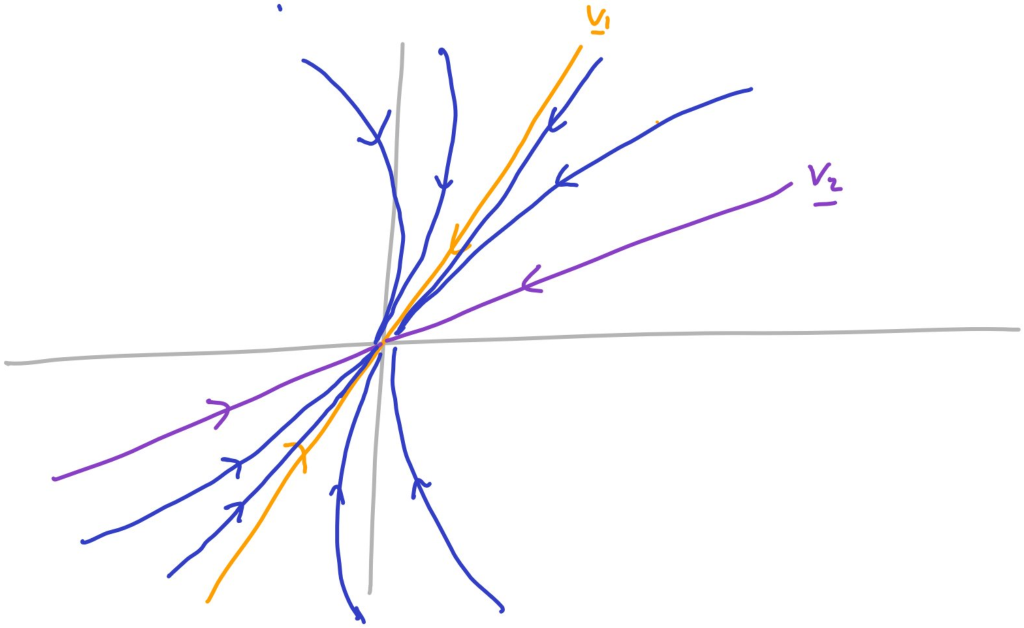
Let $\begin{pmatrix} a \\ b \end{pmatrix} = Q^{-1} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$.



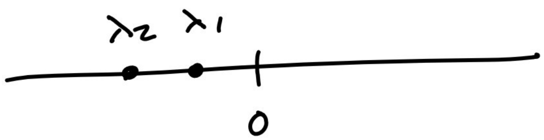




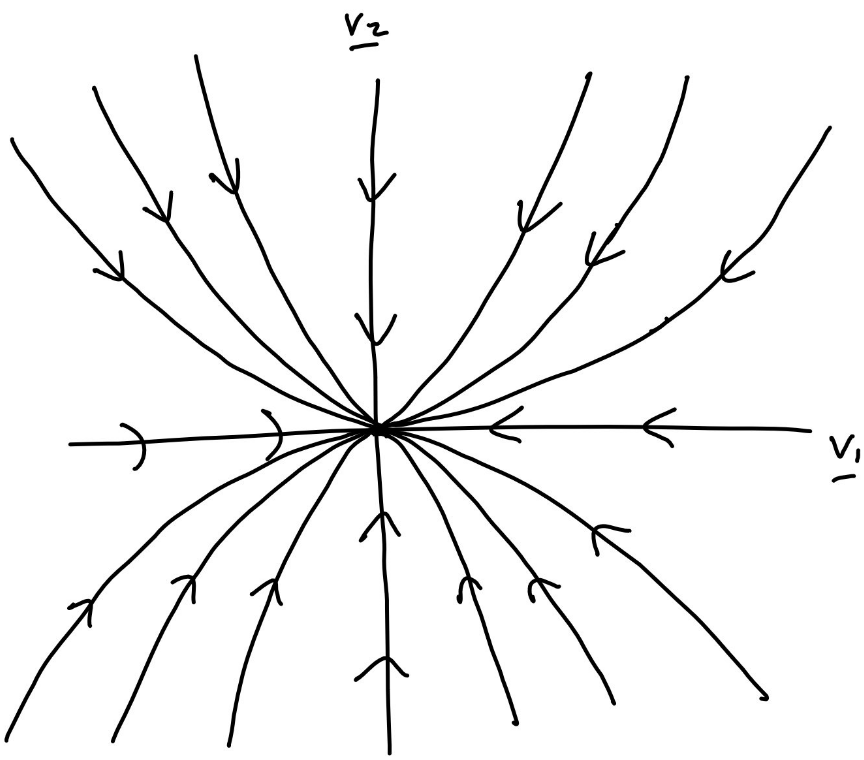
We can understand the flow lines in the $\{\underline{e}_1, \underline{e}_2\}$-coordinates as deformations of the flow lines in $\{\underline{v}_1, \underline{v}_2\}$-coordinates.



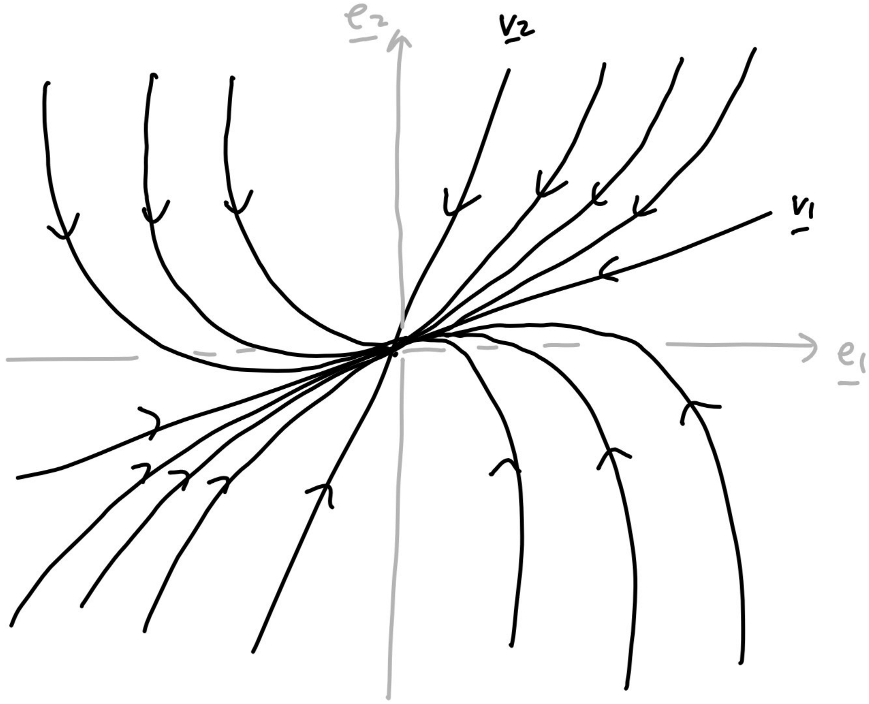
In summary, we obtain the following portrait of the flow lines / solutions (known as the phase portrait.)



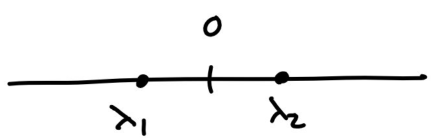
Stable node (or Sink)



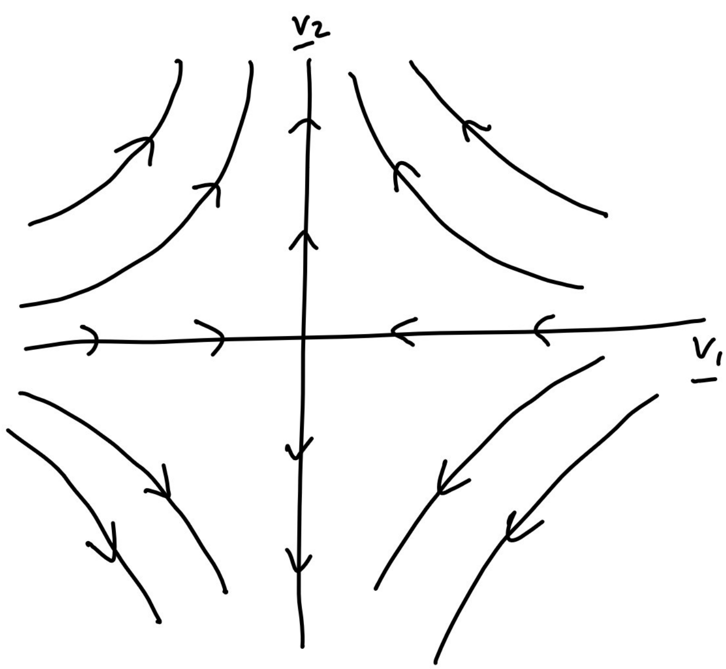
Eigenbasis

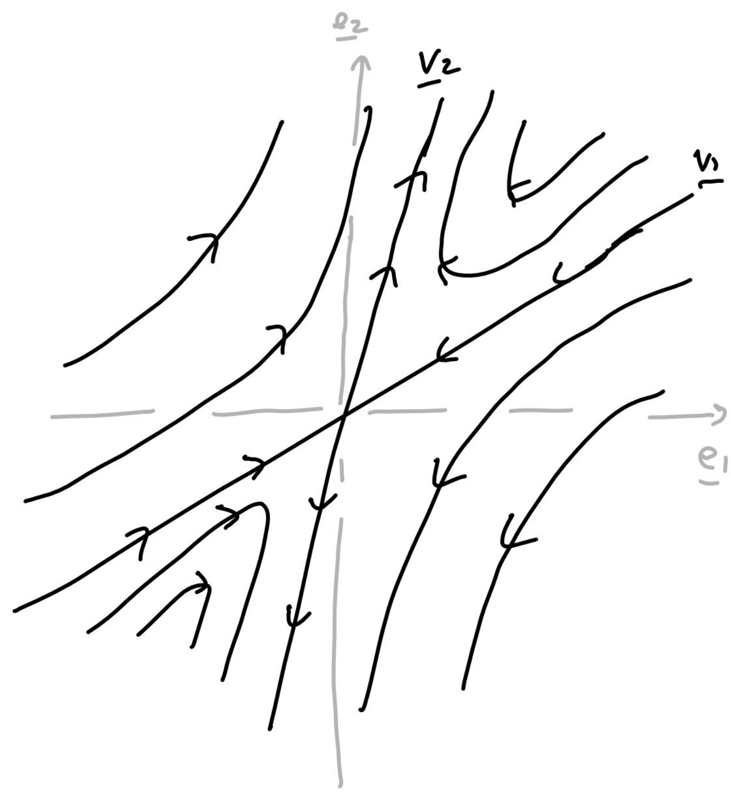


Using similar reasoning, we can find the following classification of phase portraits:



Saddle or Saddle Point

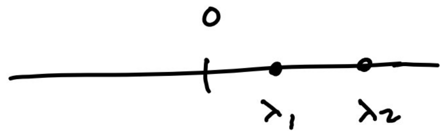


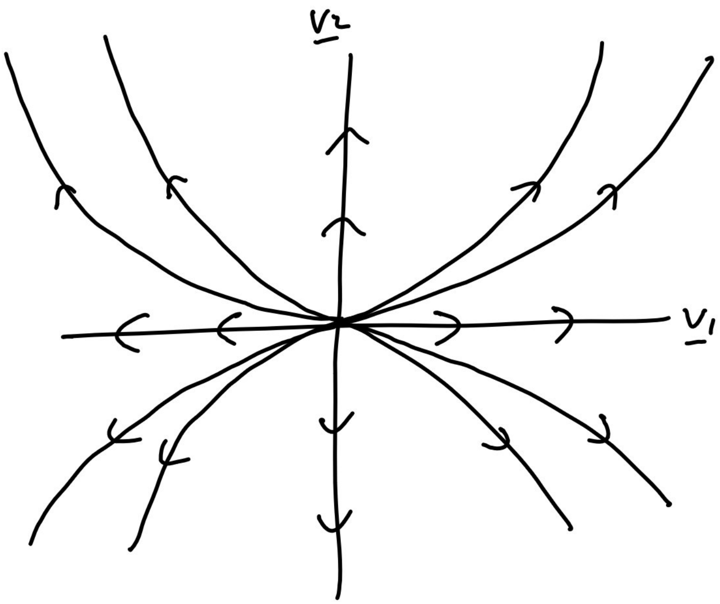


Eigenbasis

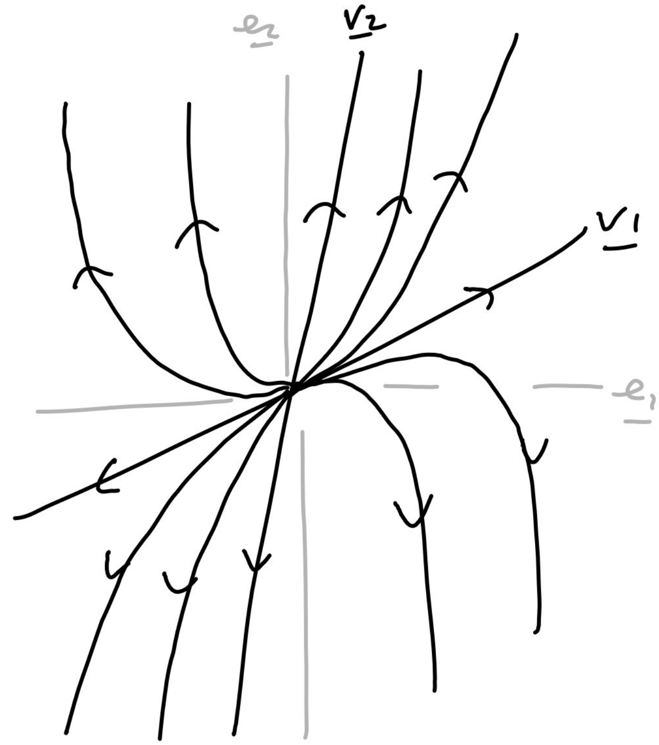
Here the flow goes <u>toward</u> the origin along one eigendirection and <u>away</u> from the origin along the other.

Unstable node (or Source)





Eigenbasis



Similar to the stable node, but flows go out from the origin as $t \to \infty$.

For complex eigenvalues, taking the real and imaginary parts of an eigenvector (call them $\underline{x}$ and $\underline{y}$, respectively) yields a basis in which the exponential is

$$\exp(At) = \begin{pmatrix} e^{\sigma t}\cos(\omega t) & e^{\sigma t}\sin(\omega t) \\ -e^{\sigma t}\sin(\omega t) & e^{\sigma t}\cos(\omega t) \end{pmatrix}$$

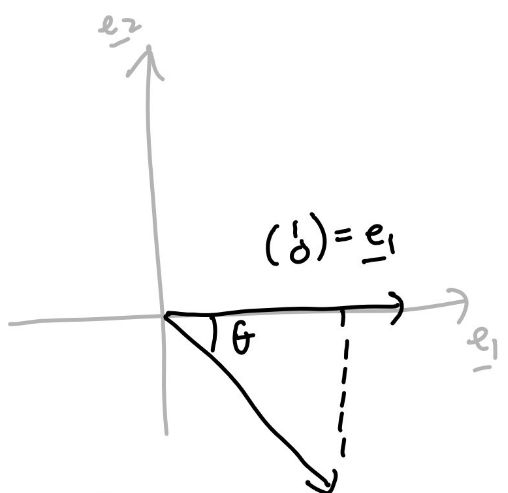
$$= e^{\sigma t}\begin{pmatrix} \cos(\omega t) & \sin(\omega t) \\ -\sin(\omega t) & \cos(\omega t) \end{pmatrix}$$

We can interpret the $e^{\sigma t}$ part as scaling the magnitude of a vector by $e^{\sigma t}$, and the

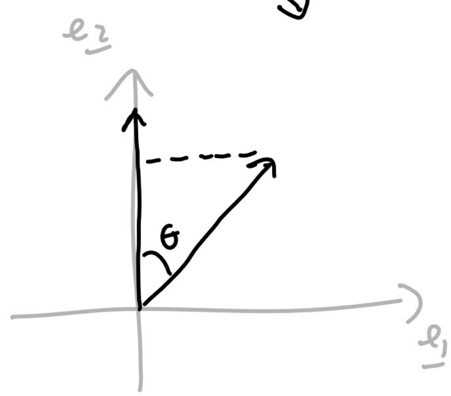
$$\begin{pmatrix} \cos(wt) & \sin(wt) \\ -\sin(wt) & \cos(wt) \end{pmatrix}$$

part as rotating a vector clockwise by wt radians.

<u>Aside</u>: What matrix in $\mathbb{R}^2$ (with respect to the basis $\underline{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\underline{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$) rotates vectors by θ radians clockwise? Call this matrix R_θ.



$$R_\theta(\underline{e}_1) = \cos(\theta)\,\underline{e}_1 - \sin(\theta)\,\underline{e}_2$$

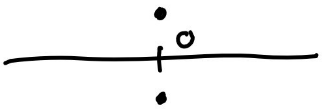


$$R_\theta(\underline{e}_2) = \sin(\theta)\,\underline{e}_1 + \cos(\theta)\,\underline{e}_2$$

Thus

$$[R_\theta]_{\{\underline{e}_1, \underline{e}_2\}} = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}.$$

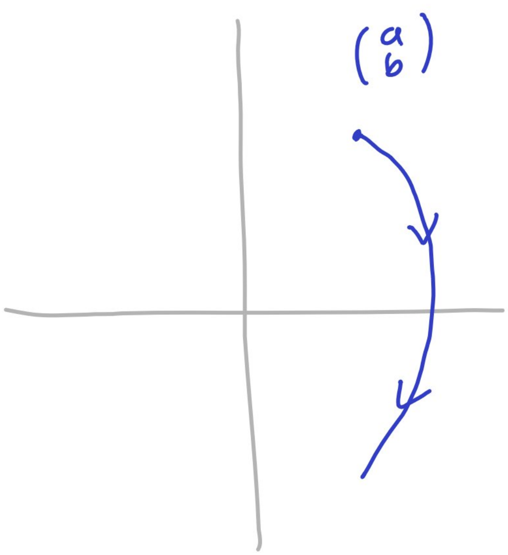
Portraits with complex eigenvalues:



(Two complex eigenvalues with no real part.) Centre $\lambda_1 = i\omega, \ \lambda_2 = -i\omega$

In the $\underbrace{\underline{x}, \underline{y}}_{\text{Real and Imaginary parts of a fixed eigenvector}}$-basis,

$$\exp(At) = \begin{pmatrix} \cos(\omega t) & \sin(\omega t) \\ -\sin(\omega t) & \cos(\omega t) \end{pmatrix}$$



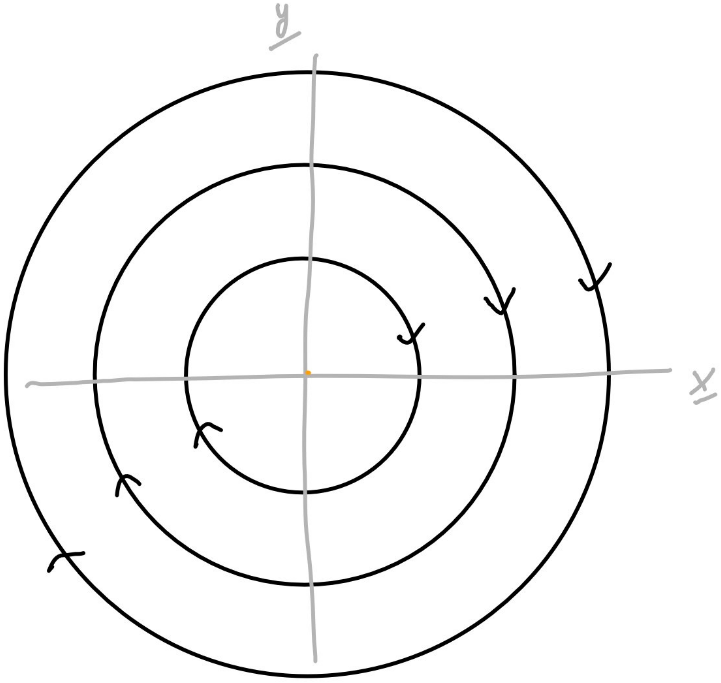
Suppose

$\underline{x}(0) = a\underline{x} + b\underline{y}$

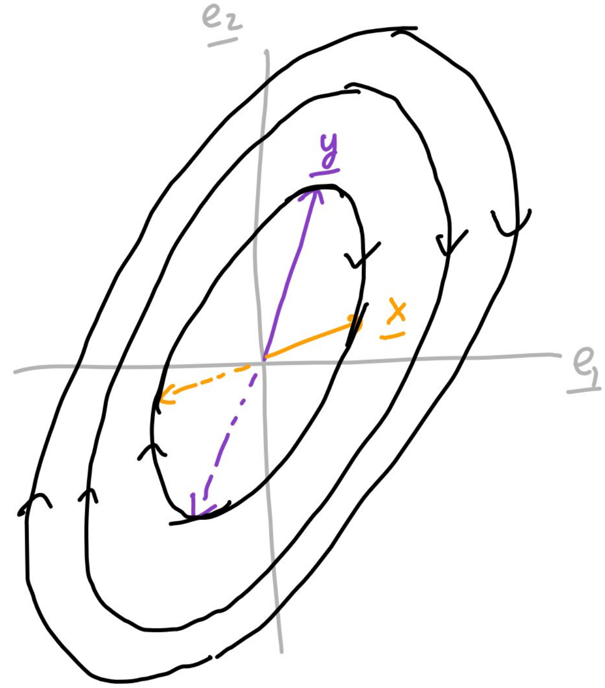
Then

$\exp(At)\,\underline{x}(0)$ rotates $\underline{x}(0)$ by ωt radians clockwise

So we see that the flow lines are circles.



$\underline{x}, \underline{y}$ - basis



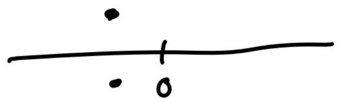
To sketch the flow in the $\underline{e}_1, \underline{e}_2$ - plane, it is helpful to notice that the flow goes from $\underline{y}$ to $\underline{x}$ to $-\underline{y}$ to $-\underline{x}$ and back to $\underline{y}$, and is parallel to $\underline{y}$ when it reaches $\underline{x}$ and parallel to $\underline{x}$ when it reaches $\underline{y}$

Portraits for λ complex with nonzero real part are similar, but with the addition of scaling

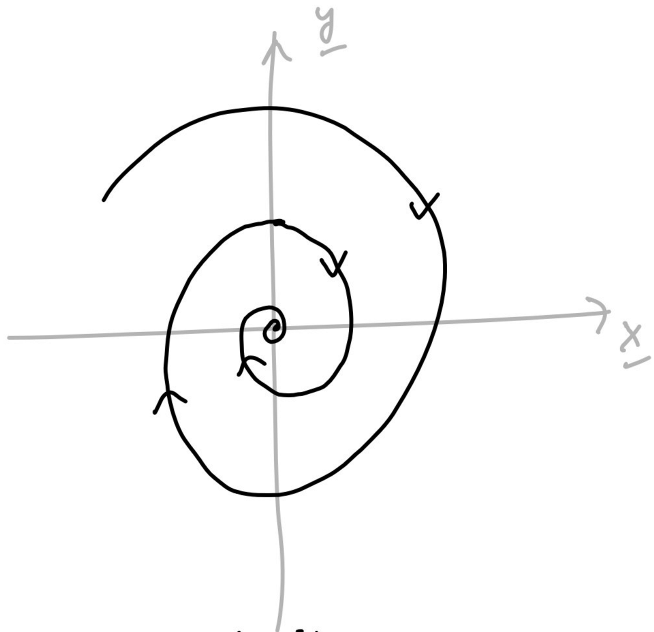
In the $\underline{x}, \underline{y}$ - basis,

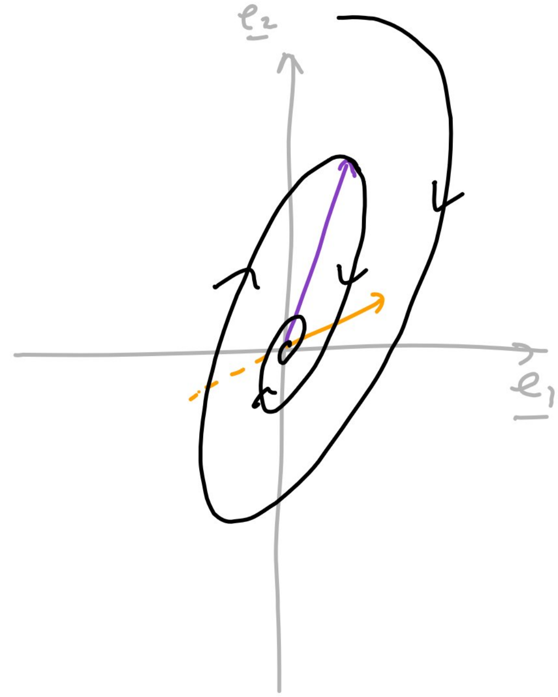
$$\exp(At) = e^{\sigma t} \begin{pmatrix} \cos(\omega t) & \sin(\omega t) \\ -\sin(\omega t) & \cos(\omega t) \end{pmatrix}$$

$\left(\text{where } \lambda = \sigma + i\omega \right)$

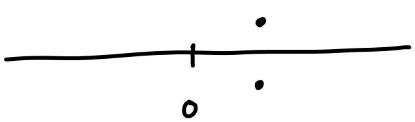


Stable spiral (Spiral sink)

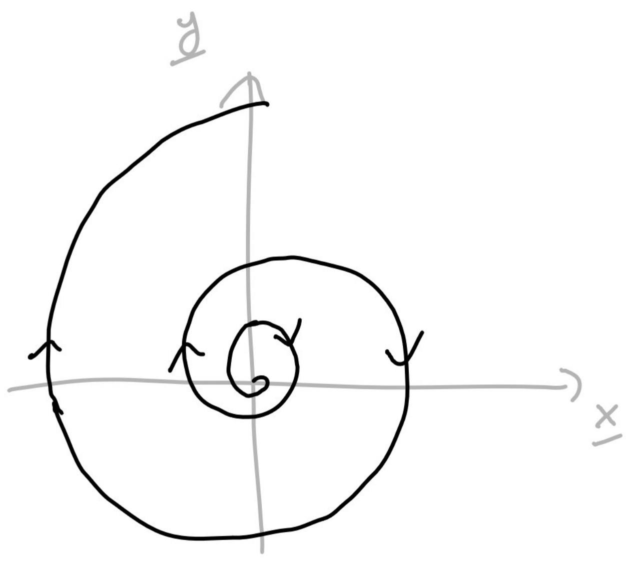


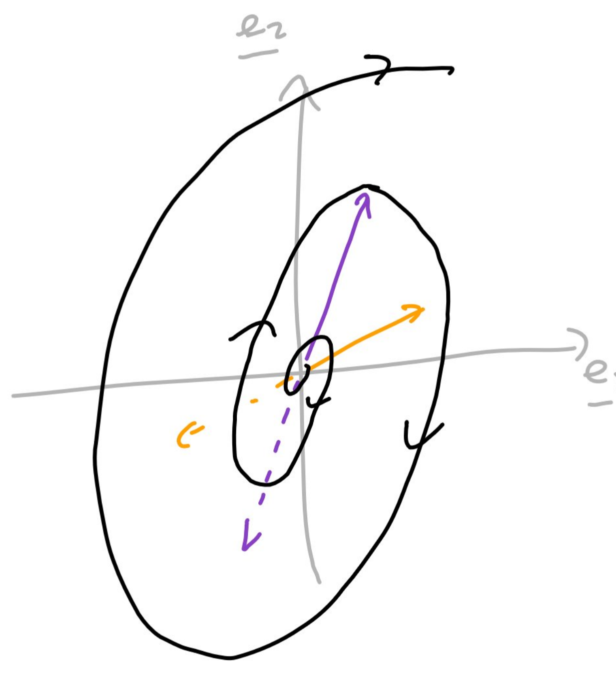


$\underline{x}, \underline{y}$ - basis



Unstable spiral (Spiral source)



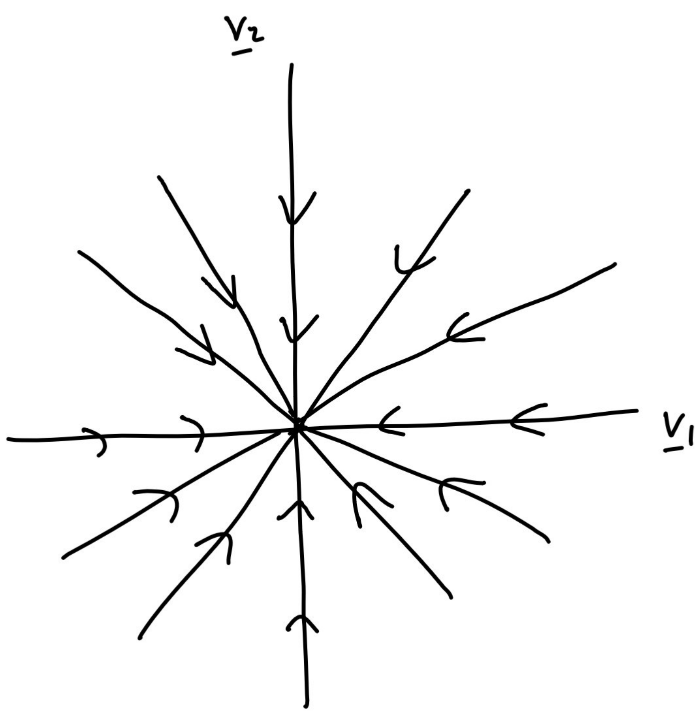


$\underline{x}, \underline{y}$ - basis

The previous five cases are the ones that occur most commonly. But for the sake of completeness, we discuss the remaining possibilities.

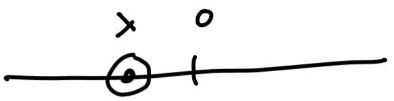
Diagonalizable

This case is similar to the stable node, but the flow lines converge to the origin at the same rate - they are actually lines.



Eigenbasis

The picture of the flow looks the same in the $\underline{e}_1, \underline{e}_2$-basis.



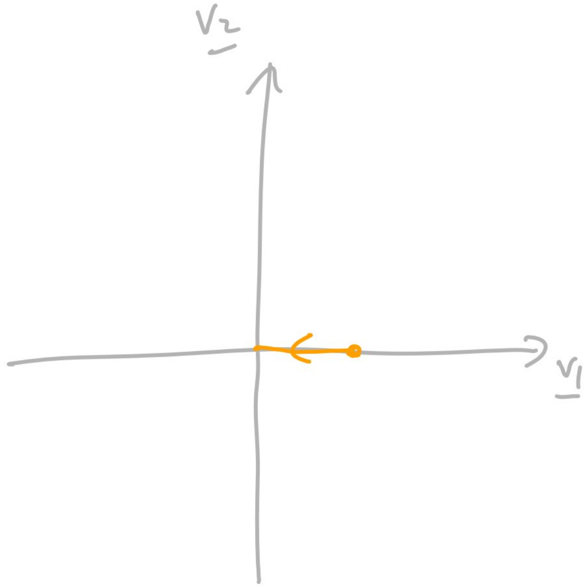
Nondiagonalizable

Let $\{\underline{v}_1, \underline{v}_2\}$ denote the Jordan canonical form basis.

In this basis,

$$\exp(At) = \begin{pmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{pmatrix}.$$

Suppose $\underline{x}(0) = a\underline{v}_1 + b\underline{v}_2$



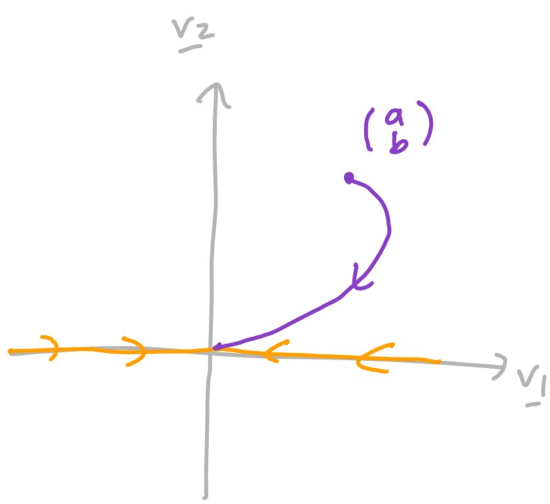
If $\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$,

$$\underline{x}(t) = \begin{pmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} e^{\lambda t} \\ 0 \end{pmatrix}$$

The flow stays on the x-axis and converges to $\underline{0}$ as $t \to \infty$ $(\lambda < 0)$

Similarly, if $\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ 0 \end{pmatrix}$.

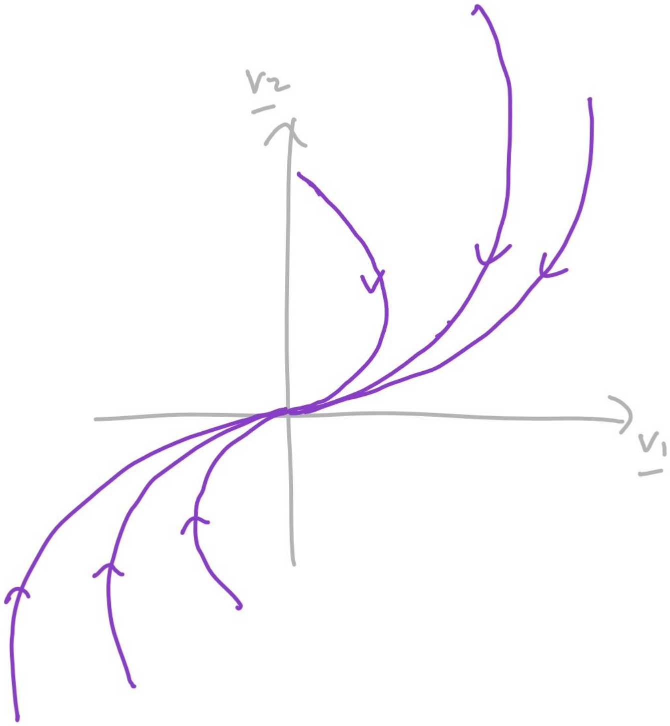


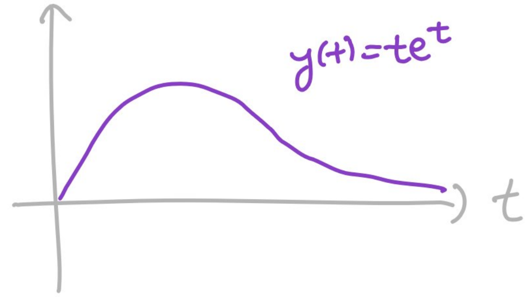
If $a > 0$, $b > 0$

$$\underline{x}(t) = \begin{pmatrix} (a+bt)e^{\lambda t} \\ be^{\lambda t} \end{pmatrix}$$

$\underline{x}(t) \to 0$ as $t \to \infty$, but the x-coordinate converges slower because of the bt term.

Near $t=0$, the x-coordinate grows because $te^{\lambda t}$ grows for a short while

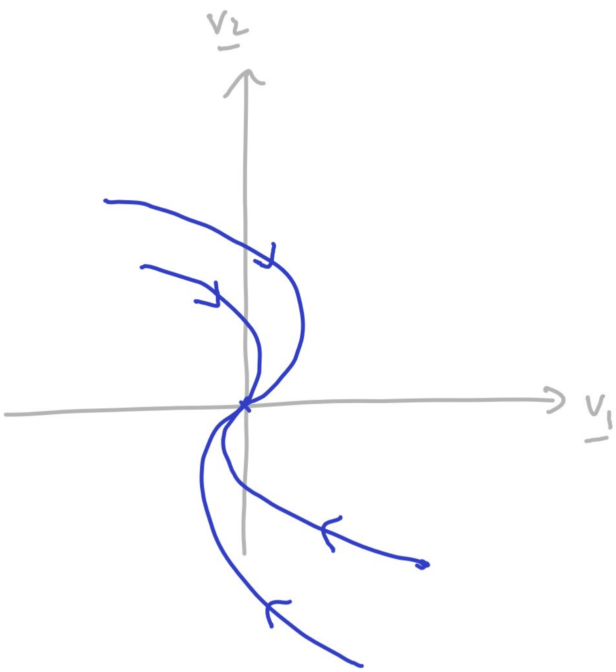




The picture for $a<0, b<0$ is similar:

$$\underline{x}(t) = \begin{pmatrix} (a+bt)e^{\lambda t} \\ be^{\lambda t} \end{pmatrix}$$

Both coefficients are now negative.



For $a>0, b<0$

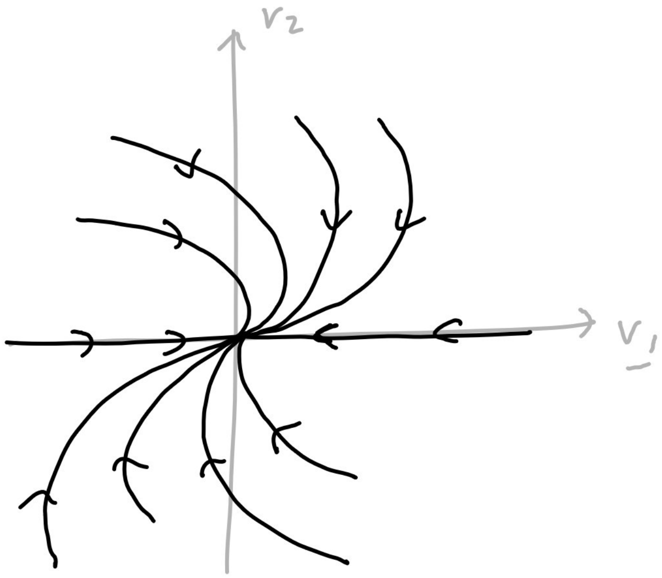
$$x(t) = \begin{pmatrix} (a+bt)e^{\lambda t} \\ be^{\lambda t} \end{pmatrix}$$

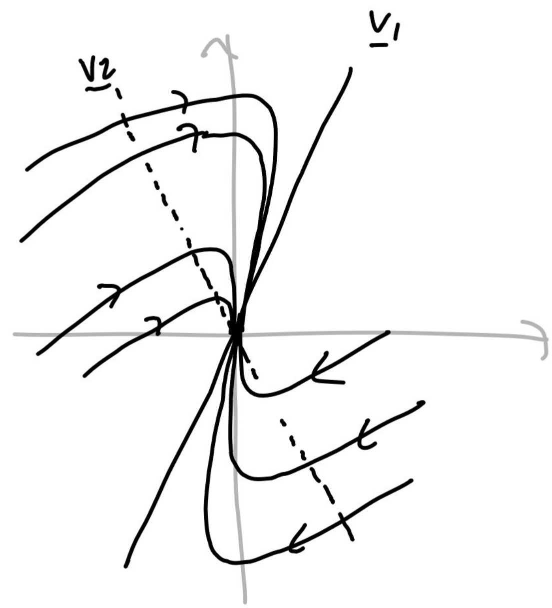
$(a+bt)$ starts positive at $t=0$ and becomes negative

For $a<0, b>0$, the picture is similar

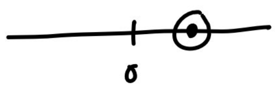
$(a+bt)$ starts negative and becomes positive.

In summary, the flow looks like

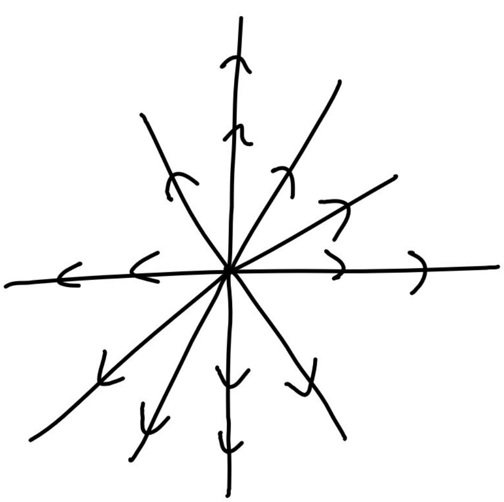




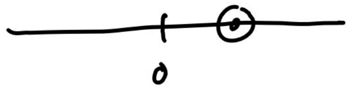
Jordan basis



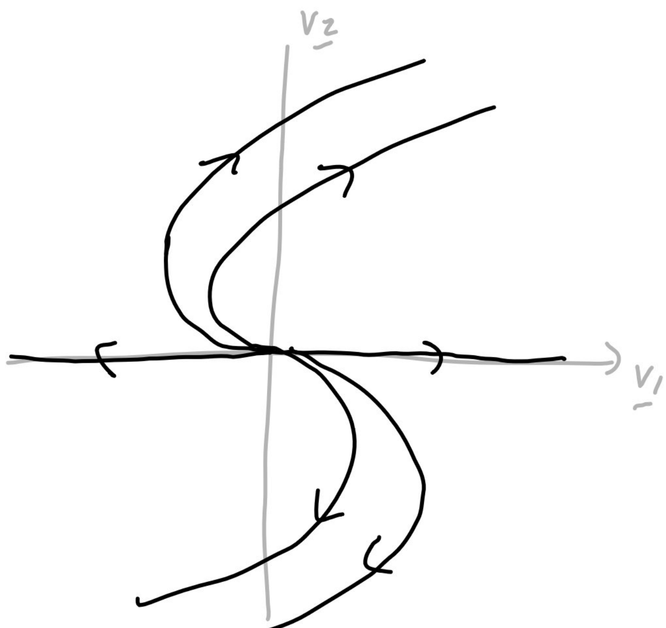
Diagonalizable



Either basis

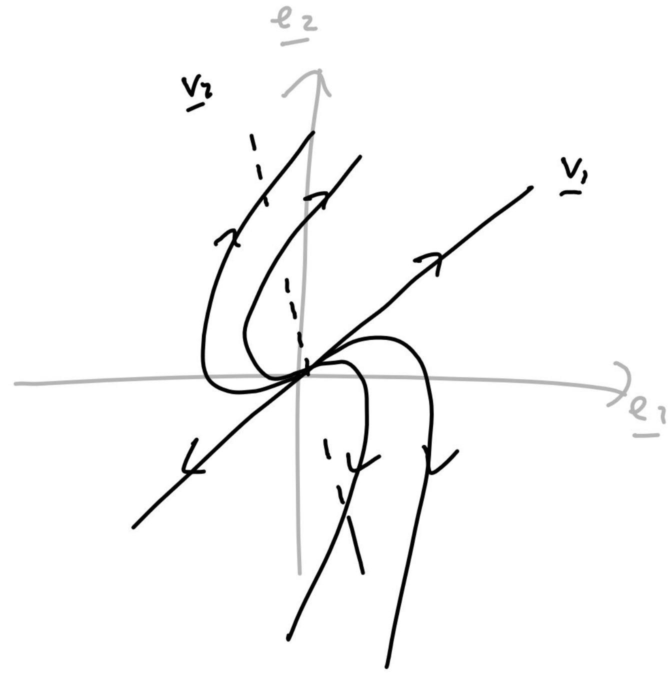


Nondiagonalizable

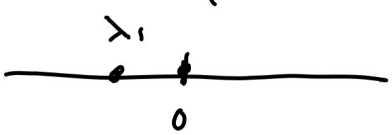


Jordan basis

Analysis is similar to the case with negative repeated eigenvalue



(zero eigenvalue)



In the eigenbasis,

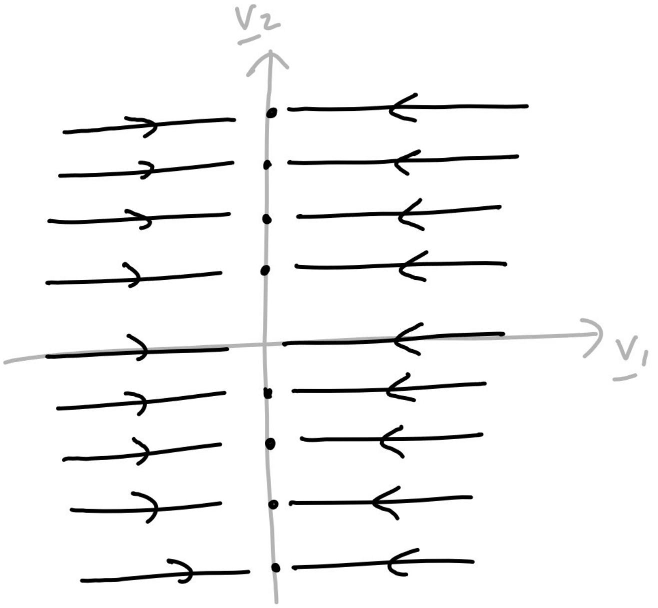
$$\exp(At) = \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & 1 \end{pmatrix}$$

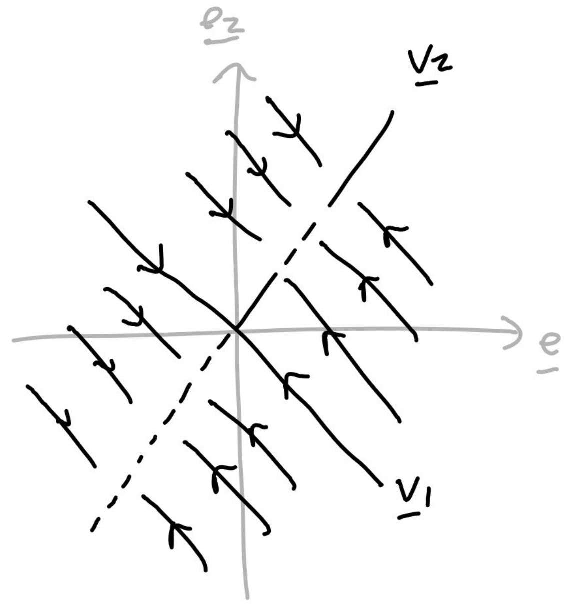
So if $\underline{x}(0) = a\underline{v}_1 + b\underline{v}_2$

$$\underline{x}(t) = a e^{\lambda_1 t}\underline{v}_1 + b\underline{v}_2$$

So the $\underline{v}_2$-coordinate is constant in the flow.

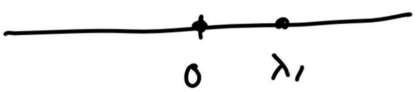
(If $a=0$, the flow stays at the same point - this is an equilibrium point.)

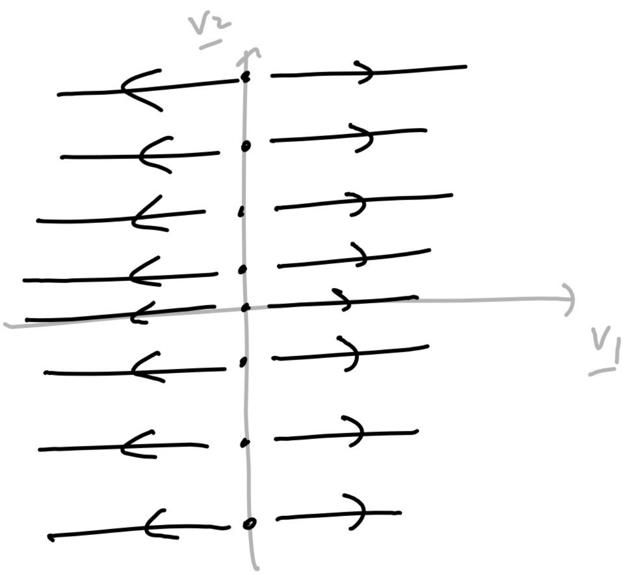


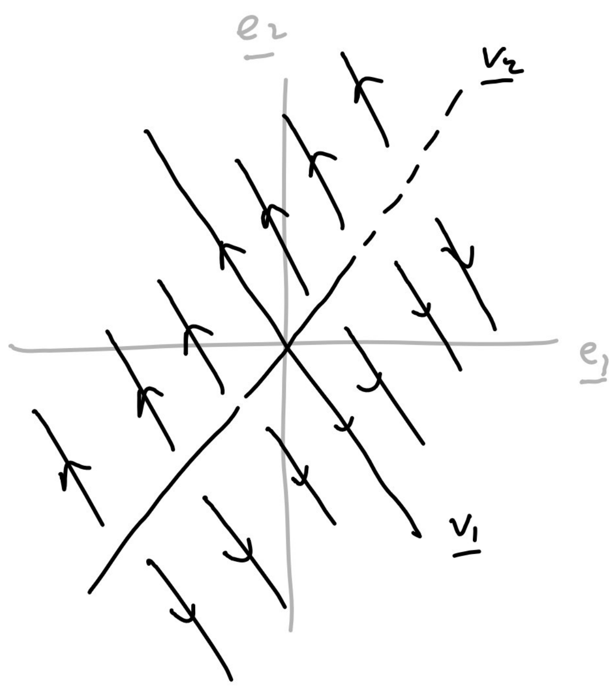


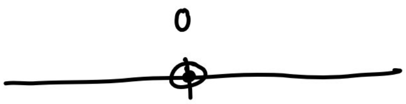
Eigenbasis

Is similar







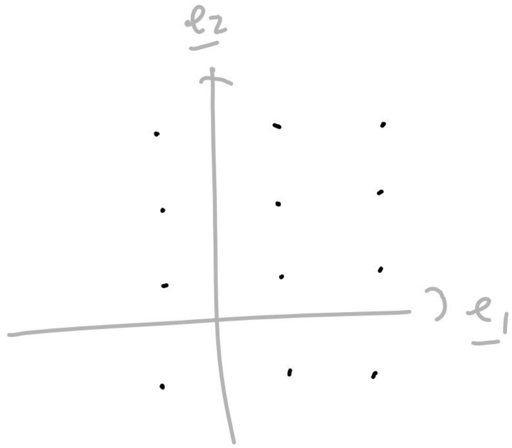


Diagonalizable

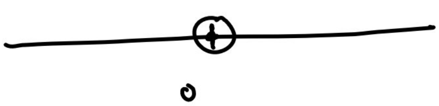
Here, in the eigenbasis,

$$\exp(At) = \begin{pmatrix} e^0 & 0 \\ 0 & e^0 \end{pmatrix} = I.$$

Every point is an equilibrium point



Every initial condition is fixed by the flow.



Nondiagonalizable

In the Jordan basis,

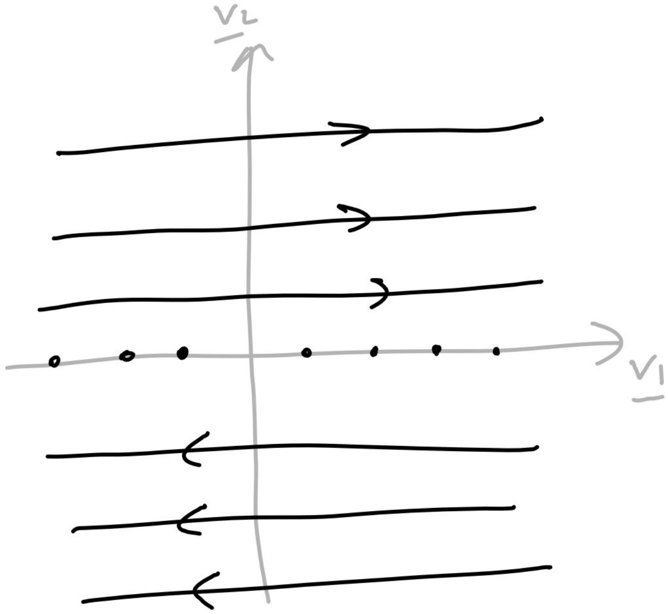
$$\exp(At) = \begin{pmatrix} e^0 & te^0 \\ 0 & e^0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

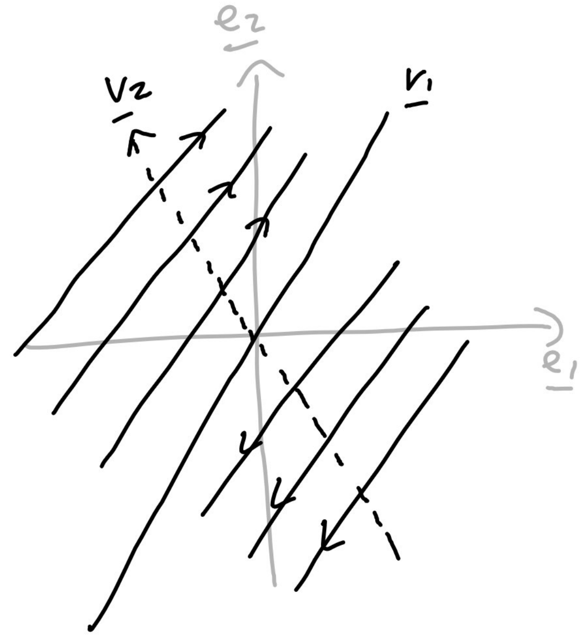
If $\underline{x}(0) = a\underline{v}_1 + b\underline{v}_2$,

$$\underline{x}(t) = \begin{pmatrix} a+bt \\ b \end{pmatrix}$$

The $\underline{v}_2$-coordinate is fixed and the direction of flow of $\underline{v}_1$-coordinate depends on the sign of b.



Jordan basis



(Points of equilibrium along span(v1))