Mthe 237

Lecture 31 (Part 2)

Nov. 21, 2017

Topic: Matrix exponential of a general real linear map

It is an (unfortunate?) fact that not every matrix can be diagonalized.

A basic example is

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

The characteristic polynomial of A is $(z-1)^2$, so A has a single eigenvalue $\lambda = 1$ of algebraic multiplicity 2.

However,

$$\begin{aligned} \ker(A - I) &= \ker \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\ &= \left\{ \begin{pmatrix} x \\ y \end{pmatrix} : \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\} \\ &= \left\{ \begin{pmatrix} x \\ y \end{pmatrix} : y = 0 \right\} \\ &= \operatorname{span} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{aligned}$$

is only one-dimensional.

We cannot find two linearly independent eigenvectors for A, and so <u>no</u> change of basis will make A a diagonal matrix.

Fortunately, there is a nice form that every matrix can be brought to by a change of basis, and it is nearly diagonal.

<u>Theorem</u>. [Existence of Jordan Canonical Form]

For any real (or complex) linear map $A: V \to V$, there is a basis of V with respect to which A takes the form

$$[A] = \begin{pmatrix} J_1 & 0 & \cdots & 0 \\ 0 & J_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \\ 0 & 0 & & J_n \end{pmatrix},$$

where

$$J_k = \begin{pmatrix} \lambda_k & 1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda_k & 1 & \cdots & 0 & 0 \\ 0 & 0 & \lambda_k & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_k & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda_k \end{pmatrix}$$ is called a Jordan block

J_k has λ_k's on the diagonal and 1's above the diagonal (and 0's elsewhere)

λ_k is one of the eigenvalues of A

If $p_A(z) = (z-\lambda_1)^{m_1} \cdots (z-\lambda_r)^{m_r}$

then the number of λ_k's on the diagonal of $\begin{pmatrix} J_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & J_n \end{pmatrix}$ is equal to m_k

Up to reordering the blocks, the Jordan Canonical Form of a matrix is unique.

Examples of Jordan canonical forms:

<u>2×2</u>: $\begin{pmatrix} \boxed{\lambda_1} & 0 \\ 0 & \boxed{\lambda_2} \end{pmatrix}$, $\begin{pmatrix} \boxed{\lambda_1} & 0 \\ 0 & \boxed{\lambda_1} \end{pmatrix}$, $\boxed{\begin{pmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{pmatrix}}$

2 Jordan blocks
$p_A(z) = (z-\lambda_1)(z-\lambda_2)$

2 Jordan blocks
$p_A(z) = (z-\lambda_1)^2$

1 Jordan block
$p_A(z) = (z-\lambda_1)^2$

<u>3×3</u>: $\begin{pmatrix} \boxed{\lambda_1} & 0 & 0 \\ 0 & \boxed{\lambda_2} & 0 \\ 0 & 0 & \boxed{\lambda_3} \end{pmatrix}$, $\begin{pmatrix} \boxed{\lambda_1} & 0 & 0 \\ 0 & \boxed{\lambda_1} & 0 \\ 0 & 0 & \boxed{\lambda_2} \end{pmatrix}$

3 Jordan blocks
$p_A(z) = (z-\lambda_1)(z-\lambda_2)(z-\lambda_3)$

3 Jordan blocks
$p_A(z) = (z-\lambda_1)^2(z-\lambda_2)$

$$\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_1 \end{pmatrix}, \quad \begin{pmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix}$$

3 Jordan blocks

$p_A(z) = (z-\lambda_1)^3$

2 Jordan blocks

$p_A(z) = (z-\lambda_1)^2(z-\lambda_2)$

$$\begin{pmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_1 \end{pmatrix}, \quad \begin{pmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 1 \\ 0 & 0 & \lambda_1 \end{pmatrix}$$

2 Jordan blocks

$p_A(z) = (z-\lambda_1)^3$

1 Jordan block

$p_A(z) = (z-\lambda_1)^3$

Any 2×2 and 3×3 matrix can be brought to one and only one of these forms by a change of basis.

Once we know how to find the matrix exponential of a matrix in Jordan Canonical Form, we'll reduce the problem of computing $\exp(A)$ to that of finding a basis that puts a matrix into Jordan Canonical Form.

Because

$$\begin{pmatrix} J_1 & 0 & \cdots & 0 \\ 0 & J_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & J_n \end{pmatrix}^p = \begin{pmatrix} J_1^p & 0 & \cdots & 0 \\ 0 & J_2^p & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & J_n^p \end{pmatrix}, \quad p \in \underset{\{0,1,2,3,\dots\}}{\underset{\shortparallel}{\mathbb{N}}}$$

we have

$$\exp\left(\begin{pmatrix} J_1 & 0 & \cdots & 0 \\ 0 & J_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & J_n \end{pmatrix}\right) = \begin{pmatrix} \exp(J_1) & 0 & \cdots & 0 \\ 0 & \exp(J_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \exp(J_n) \end{pmatrix}$$

So the problem is reduced to computing

$$\exp(J) \quad \text{for} \quad J = \begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \cdots & 0 \\ 0 & 0 & \lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda \end{pmatrix}$$

We have

$$J = \underbrace{\begin{pmatrix} \lambda & 0 & 0 & \cdots & 0 \\ 0 & \lambda & 0 & \cdots & 0 \\ 0 & 0 & \lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda \end{pmatrix}}_{} + \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

Scalar multiple of the identity, so commutes with every matrix

$$\exp(J) = \exp\begin{pmatrix} \lambda & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda \end{pmatrix} \exp\left(\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}\right)$$

$$= \begin{pmatrix} e^\lambda & 0 & 0 & \cdots & 0 \\ 0 & e^\lambda & 0 & \cdots & 0 \\ 0 & 0 & e^\lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & e^\lambda \end{pmatrix} \underbrace{\exp\left(\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}\right)}_{\text{Reduced to computing this.}}$$

$$\exp(A) = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots$$

Fortunately, powers of matrices of the last type follow a simple pattern.

With each additional power, the 1's jump "one diagonal up".

Let's illustrate this with 4×4 matrices.

Let $N = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

How does N act on the underlying basis, that we'll denote by

$\underline{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$, $\underline{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$, $\underline{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$, $\underline{e}_4 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$?

$$\left.\begin{aligned} N\underline{e}_1 &= \underline{0} \\ N\underline{e}_2 &= \underline{e}_1 \\ N\underline{e}_3 &= \underline{e}_2 \\ N\underline{e}_4 &= \underline{e}_3 \end{aligned}\right\}$$

N reduces the indices of the basis elements by one

$$0 \leftarrow \underline{e}_1 \leftarrow \underline{e}_2 \leftarrow \underline{e}_3 \leftarrow \underline{e}_4$$

It follows that

$$\begin{aligned} N^2\underline{e}_1 &= N(N\underline{e}_1) = N\underline{0} = \underline{0}, \\ N^2\underline{e}_2 &= N(N\underline{e}_2) = N\underline{e}_1 = \underline{0}, \\ N^2\underline{e}_3 &= N(N\underline{e}_3) = N\underline{e}_2 = \underline{e}_1, \\ N^2\underline{e}_4 &= N(N\underline{e}_4) = N\underline{e}_3 = \underline{e}_2. \end{aligned}$$

So N^2 reduces the indices by two, and is represented by the matrix

$$N^2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Now,

$$\left.\begin{aligned} N^3\underline{e}_1 &= N(N^2\underline{e}_1) = N\underline{0} = \underline{0}, \\ N^3\underline{e}_2 &= N(N^2\underline{e}_2) = N\underline{0} = \underline{0}, \\ N^3\underline{e}_3 &= N(N^2\underline{e}_3) = N\underline{e}_1 = \underline{0}, \\ N^3\underline{e}_4 &= N(N^2\underline{e}_4) = N\underline{e}_2 = \underline{e}_1. \end{aligned}\right\}$$

N^3 reduces indices by three

$$N^3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Finally, $N^4 \underline{e}_1 = N(N^3 \underline{e}_1) = N\underline{0} = \underline{0}$,

$N^4 \underline{e}_2 = N(N^3 \underline{e}_2) = N\underline{0} = \underline{0}$,

$N^4 \underline{e}_3 = N(N^3 \underline{e}_3) = N\underline{0} = \underline{0}$,

$N^4 \underline{e}_4 = N(N^3 \underline{e}_4) = N\underline{e}_1 = \underline{0}$.

$$N^4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and it follows that $N^m = 0$ for $m \geq 4$.

Therefore, $\exp(N) = I + N + \frac{N^2}{2!} + \frac{N^3}{3!} + \underbrace{\frac{N^4}{4!} + \cdots}_{=0}$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & \frac{1}{2!} & 0 \\ 0 & 0 & 0 & \frac{1}{2!} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & \frac{1}{3!} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 & \frac{1}{2!} & \frac{1}{3!} \\ 0 & 1 & 1 & \frac{1}{2!} \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and

$$\exp\left(\begin{pmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{pmatrix}\right) = \begin{pmatrix} e^\lambda & 0 & 0 & 0 \\ 0 & e^\lambda & 0 & 0 \\ 0 & 0 & e^\lambda & 0 \\ 0 & 0 & 0 & e^\lambda \end{pmatrix}\begin{pmatrix} 1 & 1 & \frac{1}{2} & \frac{1}{3!} \\ 0 & 1 & 1 & \frac{1}{2} \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} e^\lambda & e^\lambda & \frac{e^\lambda}{2!} & \frac{e^\lambda}{3!} \\ 0 & e^\lambda & e^\lambda & \frac{e^\lambda}{2!} \\ 0 & 0 & e^\lambda & e^\lambda \\ 0 & 0 & 0 & e^\lambda \end{pmatrix}$$

The computation for a $s \times s$ Jordan block is completely analogous, and we have

$$\exp\left(\begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \cdots & 0 \\ 0 & 0 & \lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda \end{pmatrix}\right) = e^\lambda \begin{pmatrix} 1 & 1 & \frac{1}{2!} & \cdots & \frac{1}{(s-2)!} & \frac{1}{(s-1)!} \\ 0 & 1 & 1 & \cdots & \frac{1}{(s-3)!} & \frac{1}{(s-2)!} \\ 0 & 0 & 1 & \cdots & \frac{1}{(s-4!)} & \frac{1}{(s-3)!} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

Finally, suppose that λ is complex, and we found a basis that puts A into a Jordan block for λ.

Taking the real and imaginary parts of these basis elements, we obtain a basis in which A has the form

$$\begin{pmatrix}
\sigma & \omega & 1 & 0 & & & & & \\
-\omega & \sigma & 0 & 1 & & & & & \\
& & \sigma & \omega & 1 & 0 & & & \\
& & -\omega & \sigma & 0 & 1 & & & \\
& & & & \sigma & \omega & & & \\
& & & & -\omega & \sigma & & & \\
& & & & & & \ddots & & \\
& & & & & & & 1 & 0 \\
& & & & & & & 0 & 1 \\
& & & & & & & \sigma & \omega \\
& & & & & & & -\omega & \sigma
\end{pmatrix}$$

and 0's elsewhere.

MATRIX EXPONENTIAL VIA THE JORDAN CANONICAL FORM

<u>Theorem</u> For any matrix A, there exists a real invertible matrix Q so that

$$Q^{-1} A Q = \begin{pmatrix} J_1 & & & & & \\ & \ddots & & & 0 & \\ & & J_m & & & \\ & & & K_1 & & \\ & 0 & & & \ddots & \\ & & & & & K_n \end{pmatrix}, \quad \text{where}$$

For each $1 \leq k \leq m$, $J_k = \begin{pmatrix} \lambda_k & 1 & 0 & \cdots & 0 \\ 0 & \lambda_k & 1 & \cdots & 0 \\ 0 & 0 & \lambda_k & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_k \end{pmatrix}$

is a Jordan block corresponding to one of the real eigenvalues λ_k of A

For each $1 \leq k \leq n$, $K_k = \begin{pmatrix} \sigma_k & \omega_k & 1 & 0 & & & \\ -\omega_k & \sigma_k & 0 & 1 & & & \\ & & \sigma_k & \omega_k & & & \\ & & -\omega_k & \sigma_k & & & \\ & & & & \ddots & & \\ & & & & & \sigma_k & \omega_k \\ & & & & & -\omega_k & \sigma_k \end{pmatrix}$

is a Jordan block corresponding to one of the complex eigenvalues $\lambda_k = \sigma_k + i\omega_k$ of A

We have

$$\exp(A) = Q \begin{pmatrix} \exp(J_1) & & & & & 0 \\ & \ddots & & & & \\ & & \exp(J_m) & & & \\ & & & \exp(K_1) & & \\ & 0 & & & \ddots & \\ & & & & & \exp(K_n) \end{pmatrix} Q^{-1}$$

If J_k is an $s \times s$ matrix,

$$\exp(J_k) = e^{\lambda_k} \begin{pmatrix} 1 & 1 & \frac{1}{2!} & \cdots & \frac{1}{(s-2)!} & \frac{1}{(s-1)!} \\ 0 & 1 & 1 & \cdots & \frac{1}{(s-3)!} & \frac{1}{(s-2)!} \\ 0 & 0 & 1 & \cdots & \frac{1}{(s-4)!} & \frac{1}{(s-3)!} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & & 1 & 1 \\ 0 & 0 & 0 & & 0 & 1 \end{pmatrix}$$

If K_k is a $(2s)\times(2s)$ matrix,

$$\exp(K_k) = e^{\sigma k} \left(\begin{array}{cc|cc|c|cc}
\cos(\omega k) & \sin(\omega k) & \cos(\omega k) & \sin(\omega k) & \cdots & \frac{\cos(\omega k)}{(s-1)!} & \frac{\sin(\omega k)}{(s-1)!} \\
-\sin(\omega k) & \cos(\omega k) & -\sin(\omega k) & \cos(\omega k) & & -\frac{\sin(\omega k)}{(s-1)!} & \frac{\cos(\omega k)}{(s-1)!} \\
\hline
 & & \cos(\omega k) & \sin(\omega k) & \cdots & \frac{\cos(\omega k)}{(s-2)!} & \frac{\sin(\omega k)}{(s-2)!} \\
 & & -\sin(\omega k) & \cos(\omega k) & & -\frac{\sin(\omega k)}{(s-2)!} & \frac{\cos(\omega k)}{(s-2)!} \\
\hline
 & & & & \ddots & & \vdots \\
\hline
 & 0 & & & & \cos(\omega k) & \sin(\omega k) \\
 & & & & & -\sin(\omega k) & \cos(\omega k)
\end{array}\right)$$