MTHE 237
Lecture 31
Nov. 21, 2017

Topic: Matrix exponential for Matrices with Complex and Repeated Eigenvalues

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As a reminder from linear algebra, some real matrices may not have real eigenvalues.

For example, the matrix $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ has the characteristic polynomial

$$p_A(z) = \det(A - zI) = \det\begin{pmatrix} -z & 1 \\ -1 & -z \end{pmatrix} = z^2+1,$$

which has no real roots.

However, we can always find roots of polynomials (with real (or complex) coefficients) in the complex numbers — this is the Fundamental Theorem of Algebra once again.

It turns out that the theory of eigenvalues and eigenvectors works analogously to the real case when the scalars are complex numbers.

For instance, $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ has characteristic polynomial

$$z^2+1 = (z-i)(z+i) \quad \text{(by the above computation)},$$

hence eigenvalues $\lambda_1 = i$ and $\lambda_2 = -i$.

Eigenvectors corresponding to $\lambda_1 = i$ lie in

$$\ker \begin{pmatrix} -i & 1 \\ -1 & -i \end{pmatrix}$$

$-ix + y = 0$ $\qquad \begin{pmatrix} 1 \\ i \end{pmatrix}$ is an eigenvector (corresponding to λ_1)

Similarly, we find that $\begin{pmatrix} 1 \\ -i \end{pmatrix}$ is an eigenvector corresponding to λ_2. We have the following diagonalization of A:

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \frac{1}{-2i} \begin{pmatrix} -i & -1 \\ -i & 1 \end{pmatrix}$$

There is a sublety here: the eigenvectors $\begin{pmatrix} 1 \\ i \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -i \end{pmatrix}$ no longer lie in the original vector space V, but rather a related vector space that is obtained from V by allowing complex scalars (we may denote it $\mathbb{C}V$; it is called the complexification of V, but we shall not use this term.)

If a matrix is diagonalizable with complex eigenvalues, we may compute the matrix exponential by $\exp(A) = QDQ^{-1}$, but at the cost of $\exp(A)$ now being a complex operator (with possibly complex entries in its representing matrix.)

In the setting of differential equations, it may be unclear how to interpret complex-valued solutions.

For this reason, we'll develop a way of seeing what happens in the process of passing to $\mathbb{C}V$, diagonalizing, and computing $\exp(A): \mathbb{C}V \to \mathbb{C}V$

On the level of the <u>real</u> vector space spanned by the real and imaginary parts of a complex eigenvector of A.

It turns out that this will give a way of computing $\exp(A)$ for A with complex eigenvalues <u>without</u> a passage to $\mathbb{C}V$!

Let A be a real linear map, or a matrix with real entries.

Let $\underline{v}$ be a complex eigenvector of A, with complex eigenvalue λ.

Let $\overline{\underline{v}}$ denote the complex conjugate of $\underline{v}$. It is obtained from $\underline{v}$ by taking complex conjugate of every coordinate.

Prop. Let A be real. Let λ be a complex eigenvalue of A with eigenvector $\underline{v}$. Then $\overline{\lambda}$ is also an eigenvalue of A with eigenvector $\overline{\underline{v}}$.

Proof: $\overline{A\underline{v}} = \overline{\lambda \underline{v}} = \overline{\lambda}\,\overline{\underline{v}}$. But $\overline{A\underline{v}} = A\overline{\underline{v}}$, since A has real entries. Therefore, $A\overline{\underline{v}} = \overline{\lambda}\,\overline{\underline{v}}$. ∎

Let $\underline{x} = \dfrac{\underline{v} + \overline{\underline{v}}}{2}$ and $\underline{y} = \dfrac{\underline{v} - \overline{\underline{v}}}{2i}$ be the real and complex parts of $\underline{v}$

Reminder: If $z = x+iy$,

$$\frac{z+\overline{z}}{2} = \frac{(x+iy)+(x-iy)}{2} = x = \text{Real part of } z$$

$$\frac{z-\overline{z}}{2i} = \frac{(x+iy)-(x-iy)}{2i} = y = \text{Imaginary part of } z$$

$\underline{x}$ and $\underline{y}$ have real coordinates, so they are elements of the original vector space V and not $\mathbb{C}V$.

Let $W \subset V$ be the real subspace of V spanned by $\underline{x}$ and $\underline{y}$.

$\underline{Q}$: What is the matrix that represents the restriction of A to W in the basis $\{\underline{x}, \underline{y}\}$?

$\underline{A}$: Let $\lambda = \sigma + i\omega$ (this was the eigenvalue of A)

$$A\underline{x} = A\left(\frac{\underline{v} + \overline{\underline{v}}}{2}\right) = \frac{1}{2} A\underline{v} + \frac{1}{2} A\overline{\underline{v}}$$

$$= \frac{1}{2}\lambda\underline{v} + \frac{1}{2}\overline{\lambda}\,\overline{\underline{v}}$$

$$= \frac{1}{2}\left[(\sigma + i\omega)(\underline{x} + i\underline{y}) + (\sigma - i\omega)(\underline{x} - i\underline{y})\right]$$

$$= \sigma\underline{x} - \omega\underline{y} \quad \text{(after expanding)}$$

$$A\underline{y} = A\left(\frac{\underline{v} - \overline{\underline{v}}}{2i}\right) = \frac{1}{2i}\left((\sigma + i\omega)(\underline{x} + i\underline{y}) - (\sigma - i\omega)(\underline{x} - i\underline{y})\right)$$

$$= \omega\underline{x} + \sigma\underline{y} \quad \text{(after expanding)}$$

Therefore, with respect to the basis $\{\underline{x}, \underline{y}\}$, $\underbrace{A|_W}_{\text{Restriction of } A \text{ to } W}$ is represented by the

matrix

$$A\underline{x} = \sigma \underline{x} - \omega \underline{y}$$
$$A\underline{y} = \omega \underline{x} + \sigma \underline{y}$$
$$\rightsquigarrow \quad [A|_W]_{\{\underline{x},\underline{y}\}} = \begin{pmatrix} \sigma & \omega \\ -\omega & \sigma \end{pmatrix}.$$

We'll see in a moment how to compute $\exp\begin{pmatrix} \sigma & \omega \\ -\omega & \sigma \end{pmatrix}$. This yields the following technique for complex eigenvalues:

- Find complex eigenvector $\underline{v}$
- Take the real and imaginary parts of $\underline{v}$ as basis elements
- In this basis, $[A] = \begin{pmatrix} \sigma & \omega \\ -\omega & \sigma \end{pmatrix}$ - not quite diagonal, but a form we can deal with.

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§ Computing $\exp\begin{pmatrix} \sigma & \omega \\ -\omega & \sigma \end{pmatrix}$.

We need a preliminary proposition:

Prop. Let A, B be matrices or linear maps. If $AB = BA$, then $\exp(A+B) = \exp(A)\exp(B)$.

Proof sketch:
$$\exp(A+B) = I + (A+B) + \frac{(A+B)^2}{2!} + \cdots$$
$$= I + (A+B) + \frac{(A^2 + AB + BA + B^2)}{2!} + \cdots$$

If $AB = BA$, this is

$$= I + (A+B) + \frac{(A^2 + 2AB + B^2)}{2!} + \cdots$$

On the other hand,

$$\exp(A)\exp(B) = \left(I + A + \frac{A^2}{2!} + \cdots\right)\left(I + B + \frac{B^2}{2!} + \cdots\right)$$

$$= I + (A+B) + \frac{A^2}{2!} + AB + \frac{B^2}{2!} + \cdots$$

$$= I + (A+B) + \frac{(A^2 + 2AB + B^2)}{2!} + \cdots$$

We see that if $AB = BA$, these are equal as series.

This is a proof sketch because I have ignored issues of convergence. However, this computation can be justified (you'll be able to do it after taking Real Analysis!) to give an actual proof. $\square$

If $AB \neq BA$, the proposition is <u>false</u>! In fact,

Prop. If $\exp(A+B) = \exp(A)\exp(B)$, then $AB = BA$

On HW: $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$

$AB \neq BA$, $\exp(A+B) \neq \exp(A)\exp(B)$.

Finally, we can compute $\exp\begin{pmatrix} \sigma & \omega \\ -\omega & \sigma \end{pmatrix}$.

Write $\begin{pmatrix} \sigma & \omega \\ -\omega & \sigma \end{pmatrix} = \begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix} + \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix}$.

Since $\begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix}$ is a scalar multiple of the identity matrix, it commutes with every matrix.

Applying the proposition,

$$\exp\left(\begin{pmatrix} \sigma & \omega \\ -\omega & \sigma \end{pmatrix}\right) = \exp\left(\begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix}\right)\exp\left(\begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix}\right)$$

Now $\begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix}$ is diagonal, so

$$\exp\left(\begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix}\right) = \begin{pmatrix} e^\sigma & 0 \\ 0 & e^\sigma \end{pmatrix}$$

We computed last time that

$$\exp\left(\begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix}\right) = \begin{pmatrix} \cos(\omega) & \sin(\omega) \\ -\sin(\omega) & \cos(\omega) \end{pmatrix}$$

Thus,

$$\boxed{\exp\left(\begin{pmatrix} \sigma & \omega \\ -\omega & \sigma \end{pmatrix}\right) = \begin{pmatrix} e^\sigma\cos(\omega) & e^\sigma\sin(\omega) \\ -e^\sigma\sin(\omega) & e^\sigma\cos(\omega) \end{pmatrix}}$$

Example $A = \begin{pmatrix} 3 & -2 \\ 2 & 1 \end{pmatrix}$

$$p_A(z) = \det \begin{pmatrix} 3-z & -2 \\ 2 & 1-z \end{pmatrix} = (3-z)(1-z) + 4$$
$$= z^2 - 4z + 7$$

Using the quadratic formula, the roots are

$$\frac{4 \pm \sqrt{16-28}}{2} = \frac{4 \pm \sqrt{-12}}{2} = 2 \pm \sqrt{3}\, i.$$

The eigenvalues are $\lambda_1 = 2 + \sqrt{3} i$, $\lambda_2 = \overline{\lambda_1} = 2 - \sqrt{3} i.$

Finding eigenvector corresponding to λ_1:

$$\ker \begin{pmatrix} 3-(2+\sqrt{3}i) & -2 \\ 2 & 1-(2+\sqrt{3}i) \end{pmatrix}$$

$$= \ker \begin{pmatrix} 1-\sqrt{3}i & -2 \\ 2 & -1-\sqrt{3}i \end{pmatrix}$$

Note: Multiplying first row by $\frac{1+\sqrt{3}i}{2}$ gives

$$(1-\sqrt{3}i)\left(\frac{1+\sqrt{3}i}{2}\right) = \frac{1+3}{2} = 2 \quad \text{and}$$

$$-2\left(\frac{1+\sqrt{3}i}{2}\right) = -1-\sqrt{3}i$$

$$\begin{pmatrix} 1-\sqrt{3}i & -2 \end{pmatrix} \xrightarrow[\cdot \frac{1+\sqrt{3}i}{2}]{} \begin{pmatrix} 2 & -1-\sqrt{3}i \end{pmatrix}$$

So the second row is indeed a scalar multiple of the first row.

$$(1-\sqrt{3}i)x - 2y = 0$$

$\begin{pmatrix} 2 \\ 1-\sqrt{3}i \end{pmatrix}$ is an eigenvector

real part $\swarrow$ $\qquad$ $\searrow$ imaginary part

$$\begin{pmatrix} 2 \\ 1 \end{pmatrix} \qquad \begin{pmatrix} 0 \\ -\sqrt{3} \end{pmatrix}$$

Take $\left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -\sqrt{3} \end{pmatrix} \right\}$ as a basis

$$Q = \begin{pmatrix} 2 & 0 \\ 1 & -\sqrt{3} \end{pmatrix}$$

$$Q^{-1} = -\frac{1}{2\sqrt{3}} \begin{pmatrix} -\sqrt{3} & 0 \\ -1 & 2 \end{pmatrix}$$

$\lambda = 2+\sqrt{3}i$

$$A = \begin{pmatrix} 3 & -2 \\ 2 & 1 \end{pmatrix} = Q \begin{pmatrix} 2 & \sqrt{3} \\ -\sqrt{3} & 2 \end{pmatrix} Q^{-1}$$

$$\exp\begin{pmatrix} 3 & -2 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 1 & -\sqrt{3} \end{pmatrix} \begin{pmatrix} e^2\cos(\sqrt{3}) & e^2\sin(\sqrt{3}) \\ -e^2\sin(\sqrt{3}) & e^2\cos(\sqrt{3}) \end{pmatrix} \left(-\frac{1}{2\sqrt{3}} \begin{pmatrix} -\sqrt{3} & 0 \\ -1 & 2 \end{pmatrix} \right)$$

$$= -\frac{e^2}{2\sqrt{3}} \begin{pmatrix} 2 & 0 \\ 1 & -\sqrt{3} \end{pmatrix} \begin{pmatrix} -\sqrt{3}\cos(\sqrt{3}) - \sin(\sqrt{3}) & 2\sin(\sqrt{3}) \\ \sqrt{3}\sin(\sqrt{3}) - \cos(\sqrt{3}) & 2\cos(\sqrt{3}) \end{pmatrix}$$

$$= -\frac{e^2}{2\sqrt{3}} \begin{pmatrix} -2\sqrt{3}\cos(\sqrt{3}) - 2\sin(\sqrt{3}) & 4\sin(\sqrt{3}) \\ -4\sin(\sqrt{3}) & 2\sin(\sqrt{3}) - 2\sqrt{3}\cos(\sqrt{3}) \end{pmatrix}$$

$$= e^2 \begin{pmatrix} \cos(\sqrt{3}) + \frac{\sin(\sqrt{3})}{\sqrt{3}} & -\frac{2}{\sqrt{3}}\sin(\sqrt{3}) \\ \frac{2}{\sqrt{3}}\sin(\sqrt{3}) & \cos(\sqrt{3}) - \frac{\sin(\sqrt{3})}{\sqrt{3}} \end{pmatrix}$$