MTHE 237
Lecture 30
Nov. 17, 2017

Topic: Solving Linear Homogeneous First-Order Systems, I

Some motivation:

We found that the flow lines (solutions) of the system

$$\begin{aligned}\dot{x} &= x \\ \dot{y} &= y\end{aligned}, \quad x(0) = x_0, \quad y(0) = y_0,$$

are given by $\underline{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^t \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ $(t \in \mathbb{R})$.

We can rewrite this as

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} e^t & 0 \\ 0 & e^t \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

and may be tempted to rewrite the above as

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \exp(It) \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}, \quad \text{or} \quad \underline{x}(t) = \exp(It)\, \underline{x}_0,$$

where I is the 2×2 identity matrix and $\exp(It) = \begin{pmatrix} e^t & 0 \\ 0 & e^t \end{pmatrix}$ is some kind of "matrix exponential."

(It is also worth noting the similarity of the expression $\underline{x}(t) = \exp(It)\underline{x}_0$ and the solution $x(t) = e^t x_0$ of the equation $\dot{x} = x$, $x(0) = x_0$.)

Making a leap, we may try to find a way of constructing, for any matrix (or, better, any linear map) A, a matrix (or linear map) $\exp(A)$, such that the solution of the first-order system

$$\dot{\underline{x}} = A\underline{x}, \quad \underline{x}(0) = \underline{x}_0$$

is given by

$$\underline{x}(t) = \exp(At)\,\underline{x}_0. \qquad (*)$$

In the next two lectures, we shall carry out this construction, develop a method of computing $\exp(At)$ given A, and check that $(*)$ are indeed solutions.

<u>Def.</u> For a square matrix A (or endomorphism $A: V \to V$), the matrix exponential $\exp(A)$ is defined by the series

$$\exp(A) = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots + \frac{A^n}{n!} + \cdots$$

(It is not clear at this stage whether this series converges, a question we'll return to later on.)

(Recall that the power series of the usual exponential is $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$.)

Here are two basic examples:

i) A is diagonal

$$A = \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_r \end{pmatrix}, \quad \lambda_j \in \mathbb{R}$$

Then $A^2 = \begin{pmatrix} \lambda_1^2 & & & 0 \\ & \lambda_2^2 & & \\ & & \ddots & \\ 0 & & & \lambda_r^2 \end{pmatrix}$, $A^3 = \begin{pmatrix} \lambda_1^3 & & & 0 \\ & \lambda_2^3 & & \\ & & \ddots & \\ 0 & & & \lambda_r^3 \end{pmatrix}, \ldots$

(The powers of A are again diagonal, obtained from A by taking the corresponding power of the entries.)

Then

$$\exp(A) = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots$$

$$= \begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & \ddots & 1 \end{pmatrix} + \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_r \end{pmatrix} + \begin{pmatrix} \frac{\lambda_1^2}{2!} & & 0 \\ & \ddots & \\ 0 & & \frac{\lambda_r^2}{2!} \end{pmatrix} + \begin{pmatrix} \frac{\lambda_1^3}{3!} & & 0 \\ & \ddots & \\ 0 & & \frac{\lambda_r^3}{3!} \end{pmatrix} + \cdots$$

$$= \begin{pmatrix} 1 + \lambda_1 + \frac{\lambda_1^2}{2!} + \frac{\lambda_1^3}{3!} + \cdots & & 0 \\ & \ddots & \\ 0 & & 1 + \lambda_r + \frac{\lambda_r^2}{2!} + \frac{\lambda_r^3}{3!} + \cdots \end{pmatrix}$$

$$= \begin{pmatrix} e^{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_r} \end{pmatrix}$$

ii) $A = \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix}$ $(\omega \in \mathbb{R})$.

we have $A^2 = \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix}\begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix} = \begin{pmatrix} -\omega^2 & 0 \\ 0 & -\omega^2 \end{pmatrix}$

$$= -\omega^2 \cdot I$$

$$A^3 = A^2 \cdot A = (-\omega^2 \cdot I)A = -\omega^2 \cdot A$$

$$= \begin{pmatrix} 0 & -\omega^3 \\ \omega^3 & 0 \end{pmatrix}$$

$$A^4 = A^3 \cdot A = (-\omega^2 A) \cdot A = \omega^4 I$$

$$\vdots$$

So that

$$\exp(A) = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix} + \begin{pmatrix} -\omega^2/2! & 0 \\ 0 & -\omega^2/2! \end{pmatrix} + \begin{pmatrix} 0 & -\frac{\omega^3}{3!} \\ \frac{\omega^3}{3!} & 0 \end{pmatrix} + \cdots$$

$$= \begin{pmatrix} 1 - \frac{\omega^2}{2!} + \frac{\omega^4}{4!} - \cdots & \omega - \frac{\omega^3}{3!} + \frac{\omega^5}{5!} - \cdots \\ -\left(\omega - \frac{\omega^3}{3!} + \frac{\omega^5}{5!} - \cdots\right) & 1 - \frac{\omega^2}{2!} + \frac{\omega^4}{4!} - \cdots \end{pmatrix}$$

$$= \begin{pmatrix} \cos(\omega) & \sin(\omega) \\ -\sin(\omega) & \cos(\omega) \end{pmatrix}$$

You may recognize this as the matrix that rotates vectors in $\mathbb{R}^2$ (with respect to the standard basis) by ω radians clockwise.

Aside: We obtain another representation of the complex numbers $\mathbb{C}$ by taking 2×2 matrices of the form $\begin{pmatrix} \sigma & \omega \\ -\omega & \sigma \end{pmatrix}$, $\sigma, \omega \in \mathbb{R}$

$$f : \left\{ \begin{pmatrix} \sigma & \omega \\ -\omega & \sigma \end{pmatrix} : \sigma, \omega \in \mathbb{R} \right\} \longrightarrow \mathbb{C} = \{ x + iy : x, y \in \mathbb{R} \}$$

$$\begin{pmatrix} \sigma & \omega \\ -\omega & \sigma \end{pmatrix} \longmapsto \sigma + i\omega.$$

Notice that $\begin{pmatrix} \sigma & \omega \\ -\omega & \sigma \end{pmatrix} + \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} = \begin{pmatrix} \sigma+\alpha & \omega+\beta \\ -(\omega+\beta) & \sigma+\alpha \end{pmatrix}$

So a sum of matrices of the above form is again a matrix of the same form.

Also, $\begin{pmatrix} \sigma & \omega \\ -\omega & \sigma \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} = \begin{pmatrix} \sigma\alpha - \omega\beta & \sigma\beta + \omega\alpha \\ -\omega\alpha - \sigma\beta & -\omega\beta + \sigma\alpha \end{pmatrix}$

$$= \begin{pmatrix} \sigma\alpha - \omega\beta & \sigma\beta + \omega\alpha \\ -(\sigma\beta + \omega\alpha) & \sigma\alpha - \omega\beta \end{pmatrix},$$

so product of two matrices of the above form is also a matrix of this form.

Defining $f : \left\{ \begin{pmatrix} \sigma & \omega \\ -\omega & \sigma \end{pmatrix} : \sigma, \omega \in \mathbb{R} \right\} \longrightarrow \{ x + iy : x, y \in \mathbb{R} \}$

$$\begin{pmatrix} \sigma & \omega \\ -\omega & \sigma \end{pmatrix} \longmapsto \sigma + i\omega,$$

we have

$$f\left(\begin{pmatrix} \sigma & \omega \\ -\omega & \sigma \end{pmatrix} + \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} \right) = f\left(\begin{pmatrix} \sigma + \alpha & \omega + \beta \\ -(\omega + \beta) & \sigma + \alpha \end{pmatrix} \right)$$

$$= (\sigma + \alpha) + i(\omega + \beta)$$

$$= (\sigma + i\omega) + (\alpha + i\beta)$$

$$= f\left(\begin{pmatrix} \sigma & \omega \\ -\omega & \sigma \end{pmatrix} \right) + f\left(\begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} \right)$$

$$f\left(\begin{pmatrix} \sigma & \omega \\ -\omega & \sigma \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} \right) = f\left(\begin{pmatrix} \sigma\alpha - \omega\beta & \sigma\beta + \omega\alpha \\ -(\sigma\beta + \omega\alpha) & \sigma\alpha - \omega\beta \end{pmatrix} \right)$$

$$= (\sigma\alpha - \omega\beta) + i(\sigma\beta + \omega\alpha)$$

$$= (\sigma + i\omega)(\alpha + i\beta)$$

$$= f\left(\begin{pmatrix} \sigma & \omega \\ -\omega & \sigma \end{pmatrix}\right) f\left(\begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}\right)$$

So the operations of addition and multiplication are preserved by the correspondence $\begin{pmatrix} \sigma & \omega \\ -\omega & \sigma \end{pmatrix} \leftrightarrow \sigma + i\omega$.

Moreover, the correspondence is clearly bijective.

So we can think of matrices of the form $\begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix}$ as purely real complex numbers, and

$$\exp\left(\begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix}\right) = \begin{pmatrix} e^{\sigma} & 0 \\ 0 & e^{\sigma} \end{pmatrix} \xmapsto{f} e^{\sigma} + i0$$
$$= e^{\sigma}$$

whereas $\begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix}$ correspond to purely imaginary complex numbers and

$$\exp\begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix} = \begin{pmatrix} \cos(\omega) & \sin(\omega) \\ -\sin(\omega) & \cos(\omega) \end{pmatrix} \xmapsto{f} \cos(\omega) + i\sin(\omega)$$
$$= e^{i\omega}$$

Our strategy for computing $\exp(A)$ for arbitrary matrices / linear maps A will be to reduce to the two basic cases just computed.

<u>Reminder</u>: A matrix/ linear map A is called <u>diagonalizable</u> if there exists a basis with respect to which A is represented by a diagonal matrix.

The standard way of diagonalizing matrices is to find a basis consisting of eigenvectors of the matrix.

Suppose that A and B are related by a change of basis, so that

$$A = QBQ^{-1}, \quad \text{for some change-of-basis matrix } Q.$$

Then

$$A^2 = (QBQ^{-1})(QBQ^{-1}) = QB^2Q^{-1}$$

$$A^3 = (QBQ^{-1})(QB^2Q^{-1}) = QB^3Q^{-1}$$

$$\vdots$$

$$A^n = (QBQ^{-1})(QB^{n-1}Q^{-1}) = QB^nQ^{-1}$$

$$\vdots$$

So that

$$\exp(A) = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots$$

$$= I + QBQ^{-1} + \frac{QB^2Q^{-1}}{2!} + \frac{QB^3Q^{-1}}{3!} + \cdots$$

$$= Q\left(I + B + \frac{B^2}{2!} + \frac{B^3}{3!} + \cdots\right)Q^{-1}$$

$$= Q\exp(B)\,Q^{-1}.$$

This gives rise to a technique for computing $\exp(A)$ when A is diagonalizable:

Write $A = QDQ^{-1}$, $D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_r \end{pmatrix}$

Then $\exp(A) = Q\begin{pmatrix} e^{\lambda_1} & & \\ & \ddots & \\ & & e^{\lambda_r} \end{pmatrix}Q^{-1}.$

<u>Example.</u> $A = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}.$

The eigenvalues of A can be found by finding the roots of the characteristic polynomial of A,

$$p_A(z) = \det(A - Iz) = \det\begin{pmatrix} 1-z & 2 \\ 3 & 2-z \end{pmatrix}$$

$$= (1-z)(2-z) - 6$$

$$= (2 - 3z + z^2) - 6$$
$$= z^2 - 3z - 4$$
$$= (z-4)(z+1)$$

Eigenvalues of A are: $\lambda_1 = 4$, $\lambda_2 = -1$.

Eigenvectors corresponding to λ will lie in $\ker(A - \lambda I)$.

$\boxed{\lambda_1 = 4}$ $\quad \ker\begin{pmatrix} 1-4 & 2 \\ 3 & 2-4 \end{pmatrix} = \ker\begin{pmatrix} -3 & 2 \\ 3 & -2 \end{pmatrix}$

$$-3x + 2y = 0$$

$\underline{v}_1 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ is a possible eigenvector (and the others are scalar multiples in this case.)

Check: $\begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 8 \\ 12 \end{pmatrix} = 4\begin{pmatrix} 2 \\ 3 \end{pmatrix}$

$\boxed{\lambda_2 = -1}$ $\quad \ker\begin{pmatrix} 1-(-1) & 2 \\ 3 & 2-(-1) \end{pmatrix} = \ker\begin{pmatrix} 2 & 2 \\ 3 & 3 \end{pmatrix}$

$$x + y = 0$$

$\underline{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is a possible eigenvector.

The change-of-basis matrix from the eigenbasis to the original basis is

$$Q = \begin{pmatrix} | & | \\ v_1 & v_2 \\ | & | \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 3 & -1 \end{pmatrix}$$

The inverse is $\quad Q^{-1} = -\frac{1}{5}\begin{pmatrix} -1 & -1 \\ -3 & 2 \end{pmatrix}$

$$\left(\frac{1}{\det G} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \right)$$

Therefore,

$$A = \begin{pmatrix} 2 & 1 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} 4 & 0 \\ 0 & -1 \end{pmatrix}\left(-\frac{1}{5}\begin{pmatrix} -1 & -1 \\ -3 & 2 \end{pmatrix}\right)$$

and

$$\exp(A) = \begin{pmatrix} 2 & 1 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} e^4 & 0 \\ 0 & e^{-1} \end{pmatrix}\left(-\frac{1}{5}\begin{pmatrix} -1 & -1 \\ -3 & 2 \end{pmatrix}\right)$$

$$= -\frac{1}{5}\begin{pmatrix} 2 & 1 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} -e^4 & -e^4 \\ -3e^{-1} & 2e^{-1} \end{pmatrix}$$

$$= -\frac{1}{5}\begin{pmatrix} -2e^4 - 3e^{-1} & -2e^4 + 2e^{-1} \\ -3e^4 + 3e^{-1} & -3e^4 - 2e^{-1} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{2}{5}e^4 + \frac{3}{5}e^{-1} & \frac{2}{5}e^4 - \frac{2}{5}e^{-1} \\ \frac{3}{5}e^4 - \frac{3}{5}e^{-1} & \frac{3}{5}e^4 + \frac{2}{5}e^{-1} \end{pmatrix}.$$