Mthe 237
Lecture 29
Nov. 15, 2017

Topics:
- Converting Higher-Order Equations to First-order Systems
- Basic Theory of Linear First-Order Systems

There is a nice construction that reduces a differential equation of order ≥ 2 to a system of first-order equations.

Let $r \geq 2$.

Consider the differential equation

$$\frac{d^r y}{dt^r} = F\left(y, \frac{dy}{dt}, \frac{d^2 y}{dt^2}, \dots, \frac{d^{r-1} y}{dt^{r-1}}, t\right) \qquad (*)$$

and the first-order system

$$\left.\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ &\vdots \\ \dot{x}_{r-1} &= x_r \\ \dot{x}_r &= F(x_1, x_2, \dots, x_r, t) \end{aligned}\right\} \quad (**)$$

<u>Theorem.</u> If $\varphi : I \to \mathbb{R}$ is a solution of $(*)$, then $t \mapsto \left(\varphi(t), \frac{d\varphi}{dt}(t), \dots, \frac{d^{r-1}\varphi}{dt^{r-1}}(t)\right)$ is a solution of $(**)$ over I.

If $\varphi : I \to \mathbb{R}^r$
$t \mapsto (x_1(t), x_2(t), \dots, x_r(t))$

is a solution of $(**)$, then x_1 is a solution of $(*)$ over I.

Proof: Suppose that $\varphi : I \longrightarrow \mathbb{R}$ is a solution of $(*)$.

Let
$$\left.\begin{aligned} x_1(t) &= \varphi(t) \\ x_2(t) &= \dot{\varphi}(t) \\ &\vdots \\ x_r(t) &= \varphi^{(r-1)}(t) \end{aligned}\right\} \text{ for all } t \in I.$$

We check that $t \longmapsto (x_1(t), \dots, x_r(t))$ is a solution of $(**)$ over I:

$$\dot{x}_1 = \dot{\varphi} = x_2$$
$$\dot{x}_2 = \ddot{\varphi} = x_3$$
$$\vdots$$
$$\dot{x}_{r-1} = \varphi^{(r-1)} = x_r$$
$$\dot{x}_r = \varphi^{(r)} = F(\varphi, \dot{\varphi}, \ddot{\varphi}, \dots, \varphi^{(r-1)}, t)$$

Because φ satisfies $(*)$

$$= F(x_1, x_2, \dots, x_r, t),$$

as claimed.

Now suppose $\varphi : I \longrightarrow \mathbb{R}^r$
$$t \longmapsto (x_1(t), \dots, x_r(t))$$

is a solution of $(**)$.

We check that $\varphi(t) = x_1(t)$ satisfies $(*)$.

We have
$$\dot{\varphi} = \dot{x}_1 = x_2$$
$$\ddot{\varphi} = \dot{x}_2 = x_3$$
$$\vdots$$
$$\varphi^{(r-1)} = \dot{x}_{r-1} = x_r$$
$$\varphi^{(r)} = \dot{x}_r = F(x_1, x_2, \ldots, x_r, t)$$
$$= F(\varphi, \dot{\varphi}, \ldots, \varphi^{(r-1)}, t),$$

as claimed. $\square$

Example: $$\frac{d^3y}{dt^3} + 2\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 4y = 0$$

$$\frac{d^3y}{dt^3} = -2\frac{d^2y}{dt^2} - 3\frac{dy}{dt} - 4y$$

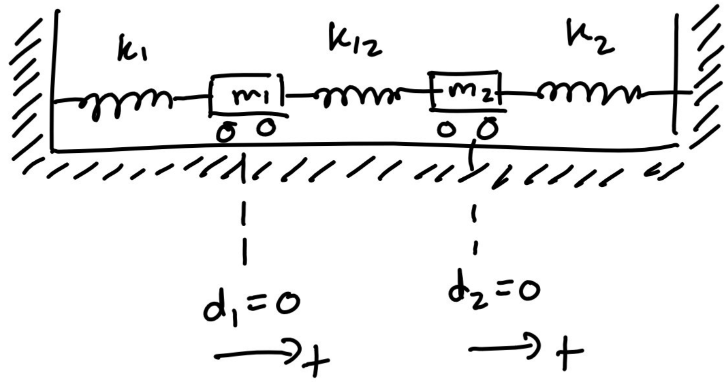
is equivalent to the first-order system

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = x_3$$
$$\dot{x}_3 = -2x_3 - 3x_2 - 4x_1$$

This, in principle, reduces the study of higher-order differential equations to the study of first-order systems.

A similar construction may be carried out for systems of higher-order equations, as in the following example:

<u>Example</u>:



Let d_1 and d_2 be displacements of m_1 and m_2 from their rest positions, respectively.

The extension of the middle spring is given by $(d_2 - d_1)$.

The equations of motion (from Newton's 2^{nd} Law) are

$$m_1 \ddot{d}_1 = -k_1 d_1 + k_{12}(d_2 - d_1)$$

$$m_2 \ddot{d}_2 = -k_2 d_2 - k_{12}(d_2 - d_1)$$

This is a system of second-order equations.

<u>Notation</u>: Write $\omega_1^2 = \dfrac{k_1}{m_1}$, $\omega_{12}^2 = \dfrac{k_{12}}{m_1}$,

$\omega_2^2 = \dfrac{k_2}{m_2}$, $\omega_{21}^2 = \dfrac{k_{12}}{m_2}$.

$$\ddot{d}_1 = -\omega_1^2 d_1 + \omega_{12}^2 (d_2 - d_1)$$
$$\ddot{d}_2 = -\omega_2^2 d_2 - \omega_{21}^2 (d_2 - d_1)$$

Rewriting once more,

$$\ddot{d}_1 = -(\omega_1^2 + \omega_{12}^2) d_1 + \omega_{12}^2 d_2$$

$$\ddot{d}_2 = \omega_{21}^2 d_1 - (\omega_{21}^2 + \omega_2^2) d_2$$

Now, to convert to a first-order system,

let x_1 "=" d_1, x_3 "=" d_2,
x_2 "=" $\dot{d}_1$, x_4 "=" $\dot{d}_2$.

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -(\omega_1^2 + \omega_{12}^2) x_1 + \omega_{12}^2 x_3$$
$$\dot{x}_3 = x_4$$
$$\dot{x}_4 = \omega_{21}^2 x_1 - (\omega_{21}^2 + \omega_2^2) x_3$$

which may be written as a matrix

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -\omega_1^2 - \omega_{12}^2 & 0 & \omega_{12}^2 & 0 \\ 0 & 0 & 0 & 1 \\ \omega_{21}^2 & 0 & -\omega_{21}^2 - \omega_2^2 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

§ Linear First-Order Systems

A first-order system of the form

$$\dot{\underline{x}}(t) = A(t)\,\underline{x}(t) + \underline{F}(t), \qquad \text{(LS)}$$

where $A(t)$ is a time-dependent linear map, is called linear. If $\underline{F}(t) = \underline{0}$, the system is called homogeneous.

If $\underline{x} \in V$, where V is an r-dimensional vector space, the system will be said to have dimension r.

With a fixed choice of basis, (LS) can be written

$$\begin{pmatrix} \dot{x}_1 \\ \vdots \\ \dot{x}_r \end{pmatrix} = \begin{pmatrix} a_{11}(t) & \cdots & a_{1r}(t) \\ \vdots & \ddots & \vdots \\ a_{r1}(t) & \cdots & a_{rr}(t) \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_r \end{pmatrix} + \begin{pmatrix} F_1 \\ \vdots \\ F_r \end{pmatrix}$$

or, expanding,

$$\dot{x}_1(t) = a_{11}(t)\,x_1(t) + \cdots + a_{1r}(t)\,x_r(t) + F_1(t)$$

$$\vdots$$

$$\dot{x}_r(t) = a_{r1}(t)\,x_1(t) + \cdots + a_{rr}(t)\,x_r(t) + F_r(t).$$

Def. A time-dependent linear map $A(t)$ is said to be continuous in t if, for a choice of basis, the entries of the matrix representing $A(t)$ in that basis are continuous functions of t.

Remark: The above definition does not depend on the choice of basis.

Theorem [Existence and Uniqueness for Linear First-Order Systems]

Suppose that $A(t)$ and $F(t)$ are continuous over an interval $I \subset \mathbb{R}$.

For any $t_0 \in I$ and $\underline{x}_0 \in \mathbb{R}^r$, the first-order system

$$\dot{\underline{x}} = A(t)\underline{x} + \underline{F}(t), \quad \underline{x} \in \mathbb{R}^r, \quad \underline{x}(t_0) = \underline{x}_0 \qquad (*)$$

has a solution $\varphi : I \longrightarrow \mathbb{R}^r$.

For any solution φ of $(*)$ over an interval J containing t_0 and contained in I, there exists a solution $\tilde{\varphi}$ over I, such that $\varphi(t) = \tilde{\varphi}(t)$ for all $t \in J$

(So that solutions can be extended to all of I.)

For any pair of solutions φ_1, φ_2 over I (in particular with $\varphi_1(t_0) = \varphi_2(t_0) = \underline{x}_0$), $\varphi_1(t) = \varphi_2(t)$ for all $t \in I$.

(So that solutions are unique.)

Theorem Suppose that $A(t)$ is continuous over an interval $I \subset \mathbb{R}$.

The set of solutions of the linear homogeneous system

$$\dot{\underline{x}} = A(t)\underline{x} \qquad x \in V \cong \mathbb{R}^r \qquad (\dot{*}*)$$

has the structure of a real vector space of dimension equal to r.

Proof (Assuming the Existence and Uniqueness Theorem)

We first show that the set of solutions of $(**)$ is a subspace of the vector space

$$C'(I, \mathbb{R}^r) = \left\{ f : I \to \mathbb{R}^r : \begin{array}{l} f \text{ and } f' \text{ exist and} \\ \text{have continuous} \\ \text{components in any} \\ \text{basis} \end{array} \right\}$$

(It is clear that it is a subset.)

It is enough to check that if φ_1 and φ_2 are solutions, then so is $c\varphi_1 + \varphi_2$, $c \in \mathbb{R}$.

$$\begin{aligned} \frac{d}{dt}(c\varphi_1 + \varphi_2) &= c\dot{\varphi}_1 + \dot{\varphi}_2 \\ &= c\left(A(t)\varphi_1\right) + \left(A(t)\varphi_2\right) \\ &= A(t)(c\varphi_1) + A(t)\varphi_2 \\ &= A(t)\left[c\varphi_1 + \varphi_2\right], \end{aligned}$$

as claimed.

Therefore, the set of solutions of $\dot{\underline{x}} = A(t)\underline{x}$ has the structure of a vector space.

To show the statement about dimension, define

$$\Psi_{t_0} : \left\{ \begin{array}{c} \text{Solutions of} \\ \dot{\underline{x}} = A(t)\underline{x} \end{array} \right\} \longrightarrow \mathbb{R}^r$$
$$\varphi \longmapsto \varphi(t_0)$$

We have

$$\begin{aligned} \Psi_{t_0}(c\varphi_1 + \varphi_2) &= (c\varphi_1 + \varphi_2)(t_0) \\ &= c\varphi_1(t_0) + \varphi_2(t_0) \\ &= c\Psi_{t_0}(\varphi_1) + \Psi_{t_0}(\varphi_2), \end{aligned}$$

so that this is a linear map.

Ψ_{t_0} is surjective by the existence part of the Existence and Uniqueness Theorem.

(Given $\underline{x}_0 \in \mathbb{R}^r$, there is a solution with initial condition $\varphi(t_0) = \underline{x}_0$, so with $\Psi_{t_0}(\varphi) = \underline{x}_0$.)

Ψ_{t_0} is injective by the uniqueness part of the Existence and Uniqueness Theorem

(If $\Psi_{t_0}(\varphi_1) = \Psi_{t_0}(\varphi_2)$, then $\varphi_1(t_0) = \varphi_2(t_0)$, so $\varphi_1 = \varphi_2$ on I by the uniqueness part of the Theorem.)

Ψ_{t_0} is a bijective linear map, and therefore an isomorphism.

So, $\dim \left\{ \begin{array}{c} \text{Solutions of} \\ \underline{\dot{x}} = A(t)\,\underline{x} \end{array} \right\} = \dim \mathbb{R}^r = r.$

$\square$