Mthe 237
Lecture 28
Nov. 14, 2017

Topic: Basic Definitions and Examples of First-Order Systems

Def. A collection of differential equations of the form

$$\left.\begin{aligned} \frac{dx_1}{dt} &= v_1(x_1, \ldots, x_r, t), \\ \frac{dx_2}{dt} &= v_2(x_1, \ldots, x_r, t), \\ &\vdots \\ \frac{dx_r}{dt} &= v_r(x_1, \ldots, x_r, t) \end{aligned}\right\} \quad (*)$$

is called a first-order system (of differential equations).

Examples:

$$\left.\begin{aligned} \frac{dx_1}{dt} &= x_1 + \cos(x_2)t^2 \\ \frac{dx_2}{dt} &= x_1 x_2 \end{aligned}\right\}, \qquad \left.\begin{aligned} \frac{dx_1}{dt} &= 3x_1 - 5x_2 \\ \frac{dx_2}{dt} &= 5x_1 t + x_2 \end{aligned}\right\}$$

Frequently, we shall use Newton's notation

$$\dot{x}_j = \frac{dx_j}{dt}$$ for the time derivative.

For $r=1,2,3$, we shall use the simpler notation

$\underline{r=1}$	$\underline{r=2}$	$\underline{r=3}$
$\dot{x} = v(x,t)$	$\dot{x} = v_1(x,y,t)$ $\dot{y} = v_2(x,y,t)$	$\dot{x} = v_1(x,y,z,t)$ $\dot{y} = v_2(x,y,z,t)$ $\dot{z} = v_3(x,y,z,t)$

<u>Def.</u> A <u>solution</u> of the first-order system $(*)$ over an interval $I \subset \mathbb{R}$ is a differentiable function

$$\varphi : I \longrightarrow \mathbb{R}^r$$
$$t \longmapsto (x_1(t), \ldots, x_r(t))$$

whose components satisfy the equations of $(*)$ for every $t \in I$.

A solution φ of $(*)$ is said to satisfy initial condition $\underline{x_0}$ at time t_0 if $\varphi(t_0) = \underline{x_0}$, where $\underline{x_0} \in \mathbb{R}^r$.

<u>Notation</u>: Underlined letters denote vectors.

$$\underline{v}, \ \underline{w}, \ \text{etc.}$$

(The usual arrow $\vec{v}$ conflicts with the dot notation for the derivative: $\dot{\vec{v}}$)

§ Geometric Interpretation of a First-Order System

<u>Reminders</u>: The velocity of a parametrized path $\underline{x} : t \mapsto (x_1(t), \dots, x_r(t))$ $(t \in I)$ is the function

$$\underline{\dot{x}} : t \longmapsto (\dot{x}_1(t), \dots, \dot{x}_r(t))$$

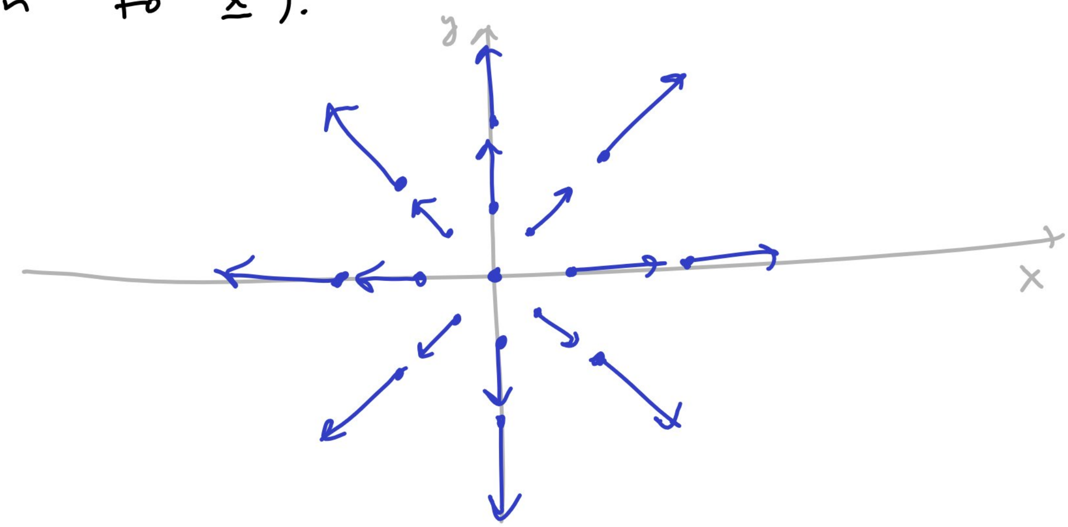
A <u>vector field</u> on $U \subset \mathbb{R}^r$ is an assignment of a vector to every point. More formally, a vector field is a function

$$\underline{v} : U \longrightarrow \mathbb{R}^r$$
$$\underline{x} \longmapsto (v_1(\underline{x}), \dots, v_r(\underline{x}))$$

Examples of vector fields: • $\underline{v} : (x, y) \mapsto (x, y)$

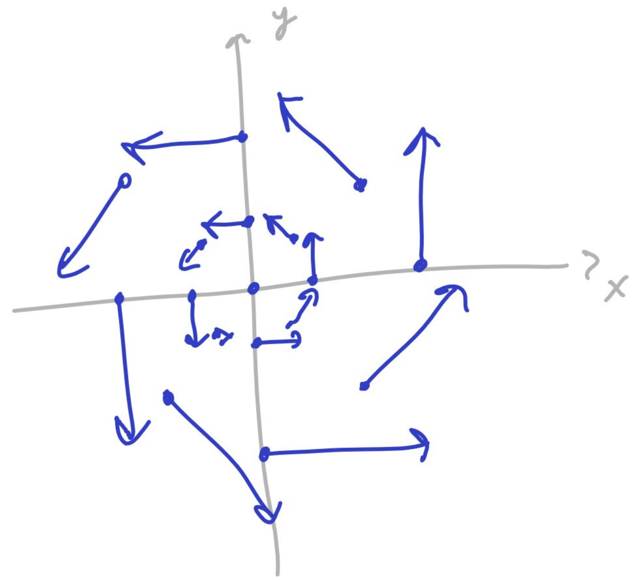
$$[\, v_1(x,y) = x, \quad v_2(x,y) = y \,]$$

To every point $\underline{x} \in \mathbb{R}^2$, $\underline{v}$ assigns the direction vector of $\underline{x}$ (the vector from the origin to $\underline{x}$).



- $\underline{v} : (x,y) \longmapsto (-y, x)$ $\left[v_1(x,y) = -y, \quad v_2(x,y) = x \right]$

Notice $(x,y)\cdot(-y,x) = 0$, so this vector is everywhere perpendicular to the position vector (in fact, to each point $\underline{v}$ assigns its position vector rotated $\frac{\pi}{2}$ radians counterclockwise.)

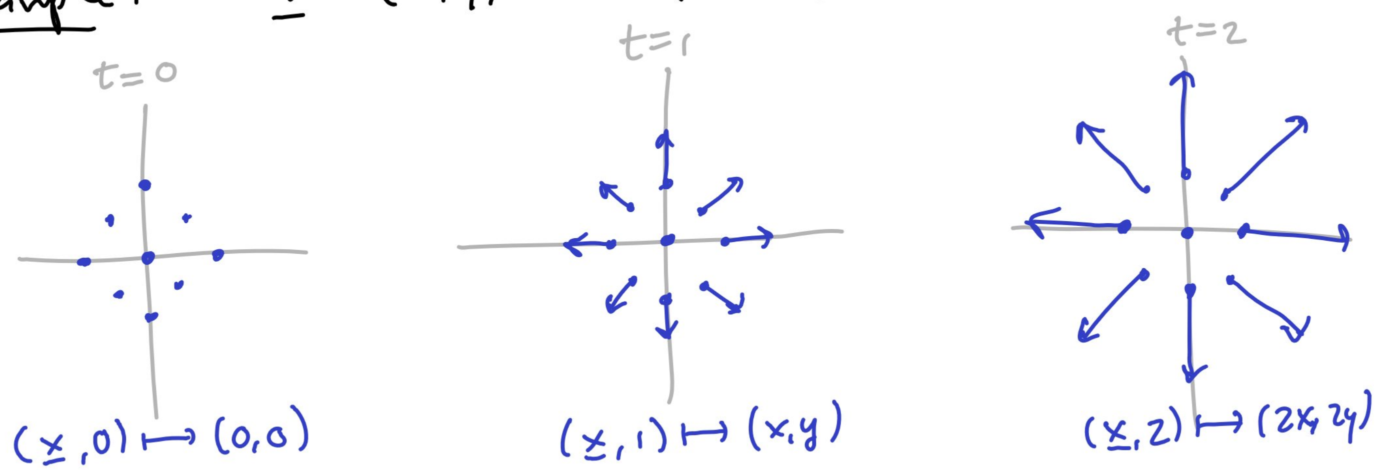


Def. A <u>time-varying vector field</u> over $U \subset \mathbb{R}^r$ and with $t \in I$ is a function

$$\underline{v} : (\underline{x}, t) \longmapsto (v_1(\underline{x},t), \dots, v_r(\underline{x},t))$$

$$\underline{x} \in U, \; t \in I$$

<u>Example</u>: $\underline{v} : (x,y,t) \longmapsto (xt, yt), \quad t \geq 0, \; \underline{x} \in \mathbb{R}^2$



Informally, thinking of a vector field as a description of flow of water (specifying the velocity of the flow at every point), a flow line is the path followed by a particle (e.g. seed) dropped into the flow.

Def. A <u>flow line</u> of a vector field $\underline{v}$ over U is a parametrized path $\underline{x} : I \longrightarrow U$ satisfying

$$\dot{\underline{x}}(t) = \underline{v}(\underline{x}(t), t) \qquad (**)$$

for all $t \in I$.

Fixing a basis of $\mathbb{R}^r$ and expanding equality $(**)$ in coordinates, we get

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_r \end{pmatrix} = \begin{pmatrix} v_1(x_1, \dots, x_r, t) \\ v_2(x_1, \dots, x_r, t) \\ \vdots \\ v_r(x_1, \dots, x_r, t) \end{pmatrix}$$

This is a first-order system! Conversely, given a first-order system, we can construct a time-varying vector field out of $v_1(\underline{x}, t), \dots, v_r(\underline{x}, t)$.

The problem of finding the flow lines of a time-varying vector field is equivalent to the problem of solving a first-order system.

Examples $\underline{v} : (x,y) \mapsto (x,y)$

Flow lines satisfy $\dot{\underline{x}}(t) = \underline{v}(\underline{x}(t),t)$.

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} , \quad \begin{aligned} \dot{x} &= x \\ \dot{y} &= y \end{aligned}$$

This system of equations is "unlinked", it is really two separate differential equations.

We can solve them to get

$$\begin{aligned} x(t) &= c_1 e^t \\ y(t) &= c_2 e^t \end{aligned} , \quad c_1, c_2 \in \mathbb{R}$$

The constants c_1, c_2 can be determined by initial conditions:

If $x(t_0) = x_0$, $y(t_0) = y_0$,

$$x(t) = x_0 e^{t-t_0}$$

$$y(t) = y_0 e^{t-t_0}$$

Writing the solution as a vector,

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{t-t_0} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

We see that flow lines are open radial rays starting from the origin.

$\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\qquad$ $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

The flow lines of $\underline{v} : (x, y, t) \mapsto (xt, yt)$ are similar.

The condition $\dot{\underline{x}}(t) = \underline{v}(\underline{x}(t), t)$ is, in coordinates,

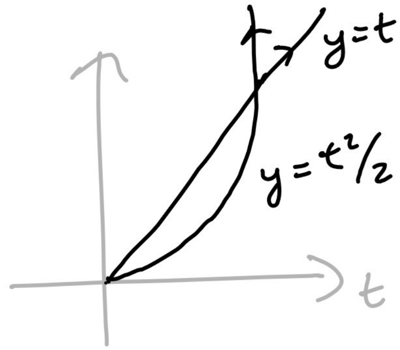
$$\dot{x} = xt$$
$$\dot{y} = yt$$

These are separable:
$$\frac{1}{x}\frac{dx}{dt} = t$$
$$\ln|x| = \frac{t^2}{2} + c_1$$
$$x(t) = c_1 e^{t^2/2}$$
$$y(t) = c_2 e^{t^2/2}$$

If $x(0) = x_0$, $y(0) = y_0$,

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{t^2/2} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

These are again radial rays, but as t increases they move at higher speed than the previous example.



$\underline{v} : (x,y) \longmapsto (-y,x)$.

Flow lines: $\dot{\underline{x}}(t) = \underline{v}(\underline{x}(t), t)$. In coordinates,

$$\dot{x} = -y$$
$$\dot{y} = x$$

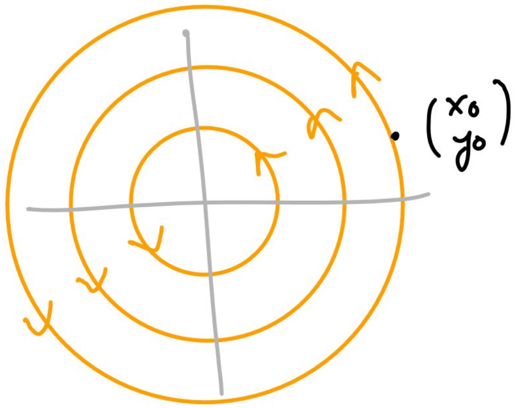
We shall learn how to solve this soon. For now, we can check that

$$t \longmapsto \big(A\cos(t+\varphi),\ A\sin(t+\varphi)\big) \quad \left(\begin{array}{c} A>0 \\ \varphi\in(\pi,\pi] \end{array}\right)$$

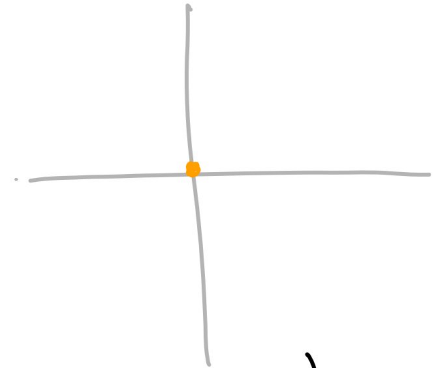
are solutions:

$$\dot{x} = \frac{d}{dt}\big(A\cos(t+\varphi)\big) = -A\sin(t+\varphi) = -y$$
$$\dot{y} = \frac{d}{dt}\big(A\sin(t+\varphi)\big) = A\cos(t+\varphi) = x$$

The flow lines are counterclockwise circles.



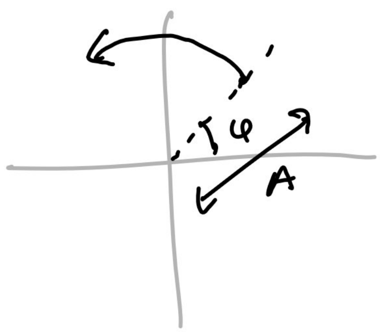
$\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$



$\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

The parameters A, φ are related to $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ by

$$A = \sqrt{x_0^2 + y_0^2}, \qquad \tan \varphi = \frac{y_0}{x_0}$$



A final example – as with single-variable first order equations, flow lines may not exist for all t:

$$\underline{v} : (x, y, t) \longmapsto (2x^2 t, 2y^2 t), \quad \begin{matrix} x(0) = 1 \\ y(0) = 1 \end{matrix}$$

Solving by separating variables,

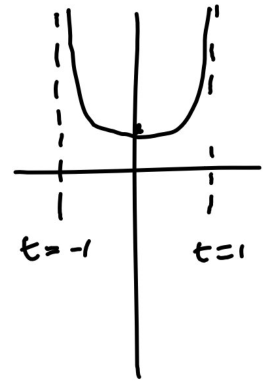
$$\dot{x} = 2x^2 t \implies \frac{1}{x^2} \frac{dx}{dt} = 2t$$

$$-\frac{1}{x} = t^2 + C$$

$$X = -\frac{1}{t^2+c}$$

$X(0)=1$, so $c=-1$

$$x(t) = \frac{1}{1-t^2}$$



Similarly, $y(t) = \dfrac{1}{1-t^2}$.

The flow lines are rays that shoot off to infinity over a finite time (with t going from 0 to 1, for instance).

Theorem [Existence and Uniqueness for First-Order Systems]

$$\dot{\underline{x}} = \underline{v}(\underline{x}, t), \qquad \underline{x}(t_0) = \underline{x}_0. \qquad (***)$$

Existence: If there exists a closed rectangle

$$\left\{ (x_1, \ldots, x_r, t) : \begin{array}{c} a_1 \le x_1 \le b_1 \\ \vdots \\ a_r \le x_r \le b_r \\ c \le t \le d \end{array} \right\} = R$$

that contains the point $(\underline{x}_0, t_0)$ in its interior, and the component functions of $\underline{v}(\underline{x}, t)$ are continuous for all $(\underline{x}, t) \in R$,

then there exists an open interval $I \subseteq \mathbb{R}$ containing t_0 and contained in $[c, d]$, and a solution $\varphi : I \longrightarrow \mathbb{R}^r$ of (***)

(In particular, $\varphi(t_0) = \underline{x}_0$.)

<u>Uniqueness</u>: If in addition the partial derivatives

$$\frac{\partial v_i}{\partial x_j}, \quad 1 \le i, j \le r$$

are continuous for all $(\underline{x}, t) \in R$, then there exists an open interval $I' \subseteq I$ containing t_0, such that if φ_1 and φ_2 are two solutions of (***) over I', then

$$\varphi_1(t) = \varphi_2(t) \quad \text{for all} \quad t \in I'.$$