Mthe 237
Lecture 27
Nov. 10, 2017

Topics: Introductions to Convolution and Green's functions

Let $f, g : \mathbb{R} \to \mathbb{R}$ be integrable (for example, continuous, or with finite number of jump discontinuities) functions.

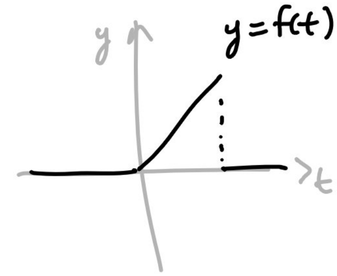
Def. The <u>convolution</u> $(f * g)$ of f and g is the function

$$(f*g)(t) = \int_{-\infty}^{\infty} f(u)\, g(t-u)\, du,$$

with domain all of t such that the integral converges.

Example:

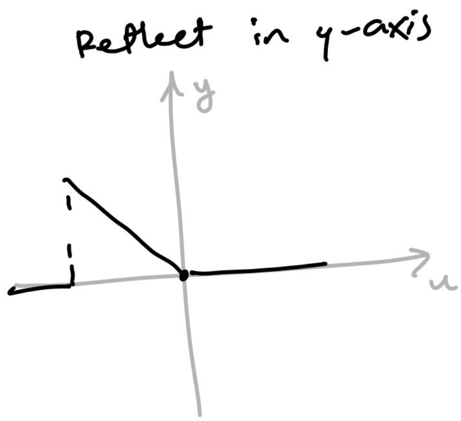
Let $f(t) = \begin{cases} t, & 0 \le t \le 1 \\ 0, & \text{otherwise} \end{cases}$

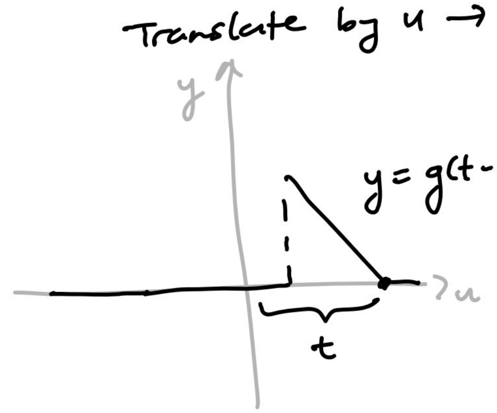


and let $g = f$.

Let's compute the convolution $(f*g)$ of f with itself (we distinguish between f and g in the notation to make the computation clearer, even though $f=g$ in this example.)

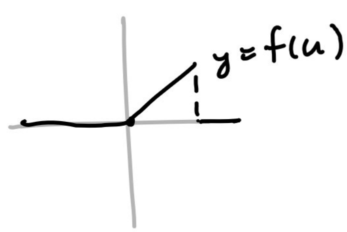
To compute $(f*g)(t)$ it is helpful to imagine reflecting the graph of g about the y-axis, and translating the result by t along the positive u direction.

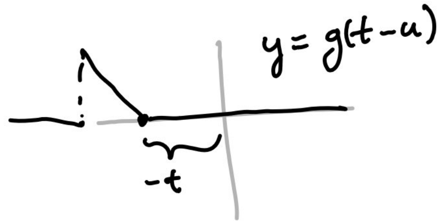




The computation of $(f*g)(t)$ in our example splits into four cases:

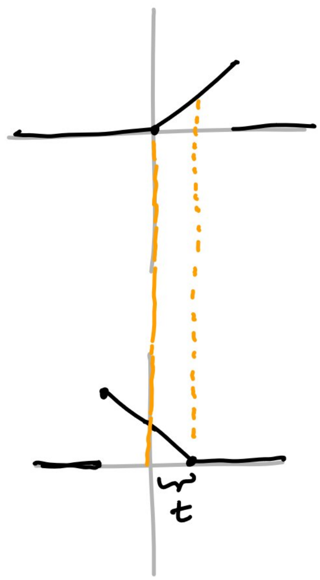
1. If $t \leq 0$, then $f(u)\,g(t-u) = 0$ for all u





So $(f*g)(t) = \int_{-\infty}^{\infty} f(u)\,g(t-u)\,du = 0$

2. If $0 < t \leq 1$, the corners of the graphs of $f(u)$ and $g(t-u)$ overlap.



$$(f*g)(t) = \int_0^t f(u)g(t-u)\,du$$

$$= \int_0^t u(t-u)\,du$$

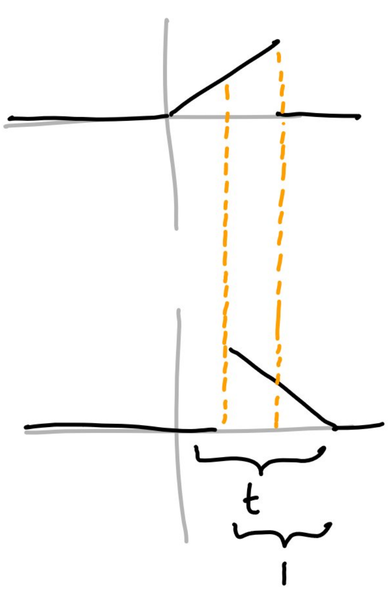
$$= \frac{t^3}{6}$$

Computation of integral: $\int_0^t u(t-u)\,du$

$$= \left[\frac{u^2 t}{2} - \frac{u^3}{3} \right]_{u=0}^{u=t}$$

$$= \left(\frac{t^3}{2} - \frac{t^3}{3} \right) - 0 = \frac{t^3}{6}$$

3. If $1 \leq t \leq 2$, $g(t-u)$ has slid past $f(u)$, but there is still some overlap



$$(f*g)(t) = \int_{t-1}^1 f(u)g(t-u)\,du$$

$$= \int_{t-1}^1 u(t-u)\,du$$

$$= \frac{t}{2} - \frac{2}{3} - \frac{t^3}{6}$$

Computation of integral: $\int_{t-1}^{1} u(t-u)\,du$

$$= \left[\frac{u^2 t}{2} - \frac{u^3}{3} \right]_{u=t-1}^{u=1}$$

$$= \left(\frac{t}{2} - \frac{1}{3} \right) - \left(\frac{(t-1)^2 t}{2} - \frac{(t-1)^3}{3} \right)$$

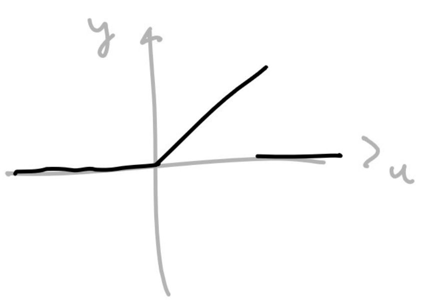
$$= \frac{t}{2} - \frac{1}{3} - \left(\frac{t^3 - 2t^2 + t}{2} - \frac{t^3 - 3t^2 + 3t - 1}{3} \right)$$

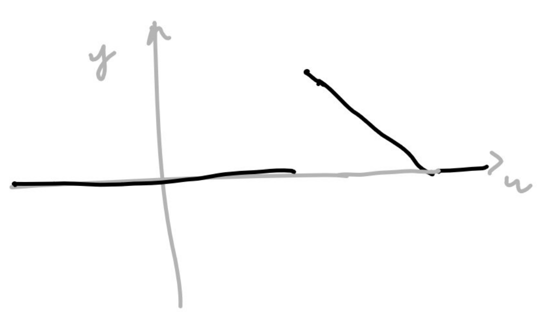
$$= \frac{t}{2} - \frac{2}{3} - \frac{t^3}{6}$$

4. Finally, if $t > 2$, $g(t-u)$ has completely slid past $f(u)$

and $f(u)g(t-u) = 0$ for all u

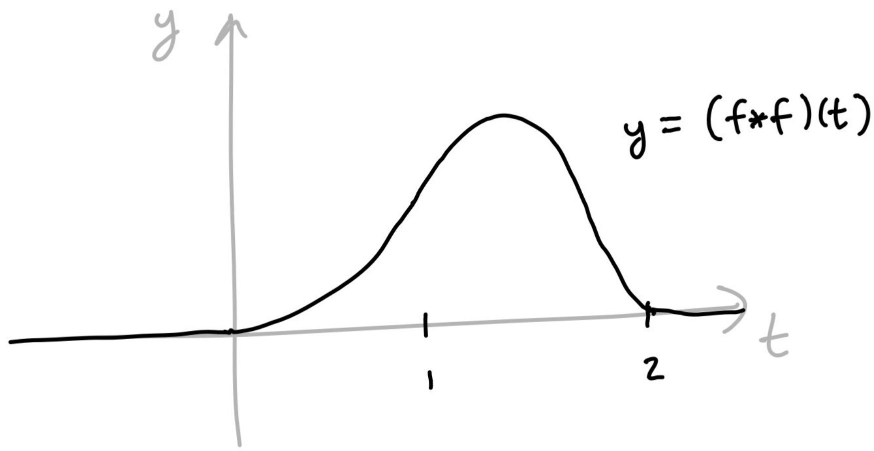
so $(f * g)(t) = 0$.





In summary,

$$(f*f)(t) = \begin{cases} t^3/6, & 0 \le t \le 1 \\ t - \frac{2}{3} - \frac{t^3}{6}, & 1 \le t \le 2 \\ 0, & \text{otherwise} \end{cases}$$



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§ Convolution and the Laplace transform

Suppose we are in the setting of the Laplace transform, working with functions

$f, g : [0,\infty) \longrightarrow \mathbb{R}$.

Extend f and g to functions defined on $\mathbb{R}$ by

$$\tilde{f}(t) = \begin{cases} f(t), & 0 \le t < \infty \\ 0, & -\infty < t < 0 \end{cases}$$

and similarly for g. From now on, we do not distinguish between f and $\tilde{f}$ explicitly.

Then $f(u) = 0$ for $u < 0$, and
for any t, $g(t-u) = 0$ for $t-u<0$, or $u>t$

Therefore, $f(u)g(t-u) = 0$ for $u \notin [0,t]$,

and we have

$$(f*g)(t) = \int_{-\infty}^{\infty} f(u)g(t-u)\,du = \int_0^t f(u)g(t-u)\,du.$$

<u>Theorem</u> Suppose that $\mathcal{L}[f](s)$ exists for $s>a$,
$\mathcal{L}[g](s)$ exists for $s>b$

(and f and g have at most finitely many jump discontinuities.)

Then $\mathcal{L}[f*g](s)$ exists for $s > \max(a,b)$, and

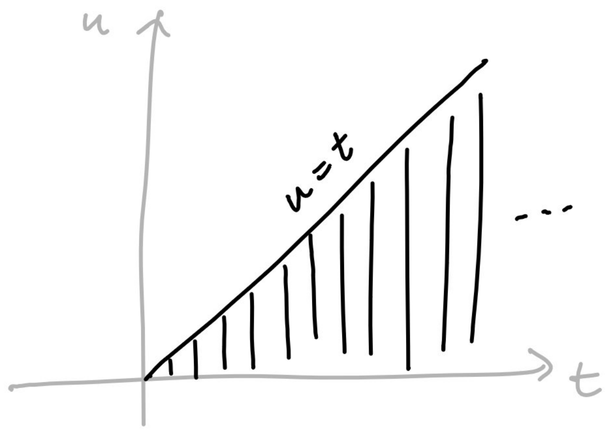
$$\mathcal{L}[f*g](s) = \mathcal{L}[f](s)\cdot \mathcal{L}[g](s) \quad \text{for all } s > \max(a,b).$$

<u>Proof</u>: The proof amounts to changing the order of integration in a double integral.

$$\mathcal{L}[f*g](s) = \int_0^{\infty} (f*g)(t)\, e^{-st}\, dt$$

$$= \int_0^{\infty} \left(\int_0^t f(u)g(t-u)\,du \right) e^{-st}\, dt$$

The region of integration is



So we can rewrite the integral as

$$\int_0^\infty \int_u^\infty f(u)\, g(t-u)\, e^{-st}\, dt\, du$$

$$= \int_0^\infty f(u) \int_u^\infty g(t-u)\, e^{-st}\, dt\, du$$

Making the change of variable $\begin{aligned} v &= t-u \\ dv &= dt \end{aligned}$

in the <u>inner</u> integral, we get

$$\int_0^\infty f(u) \int_0^\infty g(v)\, e^{-s(u+v)}\, dv\, du$$

$$= \int_0^\infty f(u)\, e^{-su} \left(\int_0^\infty g(v)\, e^{-sv}\, dv \right) du$$

$$= \left(\int_0^\infty g(v)\, e^{-sv}\, dv \right) \left(\int_0^\infty f(u)\, e^{-su}\, du \right)$$

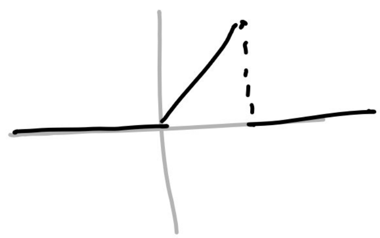
$$= \mathcal{L}[g](s) \cdot \mathcal{L}[f](s),$$

as desired.

(The hypothesis on finitely many jump discontinuities is made so that Fubini's theorem can be applied to change the order of integration.)

Let's apply this result to compute the convolution of

$$f(t) = \begin{cases} 1, & 0 \le t \le 1 \\ 0, & \text{otherwise} \end{cases}$$



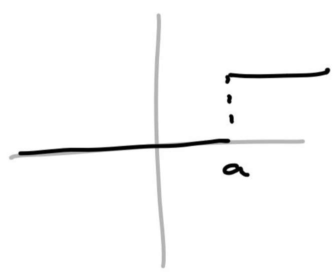
with itself in another way.

We have

$$f(t) = u_0(t) \cdot t - u_1(t) \cdot t$$

(Reminder:

$$u_a(t) = \begin{cases} 1, & t \ge a \\ 0, & t < a \end{cases}$$



Is the unit step function with jump at $t=a$.

Thus,

$$u_0(t) - u_1(t) = \begin{cases} 1, & 0 \le t \le 1 \\ 0, & \text{otherwise} \end{cases}$$
)

So $(u_0(t) - u_1(t)) \cdot t = f(t)$

Rewriting slightly, we have

$$f(t) = u_0(t)\cdot t - u_1(t)(t-1) - u_1(t)$$

<u>Recall:</u> $\mathcal{L}[u_a(t)\, g(t-a)](s) = e^{-as}\mathcal{L}[g](s)$.

So, $$\mathcal{L}[f](s) = \frac{1}{s^2} - \frac{e^{-s}}{s^2} - \frac{e^{-s}}{s}$$

By the convolution theorem,

$$\mathcal{L}[f*f](s) = \mathcal{L}[f](s)\cdot\mathcal{L}[f](s)$$

$$= \frac{1}{s^4} + \left(-\frac{2}{s^4} - \frac{2}{s^3}\right)e^{-s}$$
$$+ \left(\frac{1}{s^4} + \frac{2}{s^3} + \frac{1}{s^2}\right)e^{-2s}$$

This is just algebra:

$$\frac{1}{s^2} - \frac{e^{-s}}{s^2} - \frac{e^{-s}}{s} = \frac{1}{s^2}\left(1 - e^{-s} - se^{-s}\right).$$ Squaring, get

$$\frac{1}{s^4}\left(1-e^{-s} - se^{-s}\right)^2 = \frac{1}{s^4}\left(1 - 2e^{-s} - 2se^{-s} + e^{-2s} + 2se^{-2s} + s^2e^{-2s}\right)$$

Taking the inverse Laplace transform, we get

(Using the facts
$$\mathcal{L}[t^3](s) = \frac{3!}{s^4} = \frac{6}{s^4}, \quad \mathcal{L}[t^2](s) = \frac{2}{s^3}, \quad \mathcal{L}[t](s) = \frac{1}{s^2}$$
)

$$(f * f)(t) = \frac{t^3}{6} + \left(-\frac{2(t-1)^3}{6} - \frac{2(t-1)^2}{2}\right) u_1(t)$$
$$+ \left(\frac{(t-2)^3}{6} + \frac{2(t-2)^2}{2} + (t-2)\right) u_2(t)$$

$$= \frac{t^3}{6} + \left(-\frac{t^3}{3} + t - \frac{2}{3}\right) u_1(t) + \left(\frac{t^3}{6} - t + \frac{2}{3}\right) u_2(t)$$

$$= \begin{cases} \frac{t^3}{6}, & 0 \le t < 1 \\ -\frac{t^3}{6} + t - \frac{2}{3}, & 1 \le t < 2 \\ 0, & t > 2 \end{cases}$$

Just as computed using the integral definition.

A Short Introduction To Green's Functions

Suppose that we would like to solve

$$(*)\begin{cases} \dfrac{d^r y}{dt^r} + a_{r-1}\dfrac{d^{r-1}y}{dt^{r-1}} + \cdots + a_1\dfrac{dy}{dt} + a_0 y = F(t), \quad a_j \in \mathbb{R} \\ \text{with initial conditions} \\ y(0)=0, \quad \dfrac{dy}{dt}(0)=0, \ldots, \dfrac{d^{r-1}y}{dt^{r-1}}(0)=0. \end{cases}$$

(and where F is of exponential order.)

Taking the Laplace transform of both sides, we get

$$s^r \mathcal{L}[y](s) + a_{r-1}s^{r-1}\mathcal{L}[y](s) + \cdots + a_1 s\mathcal{L}[y](s) + a_0\mathcal{L}[y](s) = \mathcal{L}[F](s),$$

$$(s^r + a_{r-1}s^{r-1} + \cdots + a_1 s + a_0)\mathcal{L}[y](s) = \mathcal{L}[F](s)$$

$$\chi(s)\,\mathcal{L}[y](s) = \mathcal{L}[F](s),$$

where $\chi(z)$ is the characteristic polynomial of the associated homogeneous equation.

Thus,

$$\mathcal{L}[y](s) = \left(\frac{1}{\chi(s)}\right)\mathcal{L}[F](s).$$

Let $G(s)$ denote $\frac{1}{\chi(s)}$ and

$g(t) = \mathcal{L}^{-1}[G]$

Then by the convolution theorem,

$$y(t) = (F * g)(t) = \int_0^t F(u)\, g(t-u)\, du.$$

We have shown that the solution of (*) with <u>any</u> forcing term F is given by the convolution $(F * g)(t)$ of F with the function g!

<u>Terminology</u>: The function g is called <u>Green's function</u>, and its Laplace transform G the <u>transfer function</u> of the equation

$$\frac{d^r y}{dt^r} + a_{r-1}\frac{d^{r-1} y}{dt^{r-1}} + \cdots + a_1 \frac{dy}{dt} + a_0 y = 0$$

<u>Examples of Green's functions</u>:

<u>First order</u>:

$$\frac{dy}{dt} + ay = 0 \quad , \qquad y(0) = 0$$

The characteristic polynomial is

$$\chi(s) = s + a.$$

Then, $G(s) = \frac{1}{s+a}$ and so $\boxed{g(t) = e^{-at}.}$

The solution to

$$\frac{dy}{dt} + ay = F(t), \quad y(0) = 0$$

is given by

$$y(t) = \int_0^t F(u)\, e^{-a(t-u)}\, du$$

$$= e^{-at} \int_0^t F(u)\, e^{au}\, du$$

This is the same solution as one we found using variation of parameters!

Second-order

$$\frac{d^2y}{dt^2} + 2\gamma \frac{dy}{dt} + \omega_0^2 y = 0, \quad y(0) = 0, \quad \frac{dy}{dt}(0) = 0.$$

The characteristic polynomial is

$$\chi(s) = s^2 + 2\gamma s + \omega_0^2.$$

Green's function of the equation depends on the roots of $\chi(s)$.

$$G(s) = \frac{1}{s^2 + 2\gamma s + \omega_0^2}$$

Let r_1 and r_2 be the roots of $\chi(s)$ (possibly equal or complex). We have

$$(s - r_1)(s - r_2) = s^2 - (r_1 + r_2)s + r_1 r_2 .$$

So, $$\left.\begin{aligned} 2\gamma &= -(r_1 + r_2) \\ \omega_0^2 &= r_1 r_2 \end{aligned}\right\}$$

Moreover, $$r_1 - r_2 = \left(\frac{-2\gamma + 2\sqrt{\gamma^2 - \omega_0^2}}{2} \right) - \left(\frac{-2\gamma - 2\sqrt{\gamma^2 - \omega_0^2}}{2} \right)$$
$$= 2\sqrt{\gamma^2 - \omega_0^2}$$

Two distinct real roots:

$$\frac{1}{(s - r_1)(s - r_2)} = \frac{1}{r_1 - r_2}\left(\frac{1}{s - r_1} - \frac{1}{s - r_2} \right)$$

So $$g(t) = \frac{1}{r_1 - r_2}\left(e^{r_1 t} - e^{r_2 t} \right)$$

$$= \frac{1}{r_1 - r_2} e^{\frac{(r_1 + r_2)}{2} t} \left(e^{\frac{r_1 - r_2}{2} t} - e^{-\left(\frac{r_1 - r_2}{2}\right) t} \right)$$

$$= \frac{2}{r_1 - r_2} e^{\left(\frac{r_1+r_2}{2}\right)t} \sinh\left(\frac{r_1 - r_2}{2} t\right).$$

Rewriting in terms of the coefficients, this is

$$\frac{e^{-\gamma t}}{\sqrt{\gamma^2 - \omega_0^2}} \sinh\left(\sqrt{\gamma^2 - \omega_0^2}\, t\right)$$

Writing $\sqrt{\gamma^2 - \omega_0^2} = \alpha$,

$$g(t) = e^{-\gamma t} \frac{\sinh(\alpha t)}{\alpha}$$

Repeated real root:

$$G(s) = \frac{1}{(s-r)^2}, \qquad g(t) = t e^{rt}.$$

Rewriting in terms of coefficients,

$$g(t) = t e^{-\gamma t}$$

Complex conjugate roots:

$$G(s) = \frac{1}{s^2 + 2\gamma s + \omega_0^2} = \frac{1}{(s+\gamma)^2 - \gamma^2 + \omega_0^2}$$

$$= \frac{1}{\sqrt{\omega_0^2 - \gamma^2}} \frac{\sqrt{\omega_0^2 - \gamma^2}}{(s+\gamma)^2 - (\omega_0^2 - \gamma^2)}$$

Since $\mathcal{L}\left[e^{\sigma t}\sin(\omega t)\right](s) = \dfrac{\omega}{(s-\sigma)^2+\omega^2}$, we see that

$$g(t) = \frac{1}{\sqrt{\omega_0^2-\gamma^2}}\, e^{-\gamma t} \sin\left(\sqrt{\omega_0^2-\gamma^2}\, t\right)$$

Writing $\omega = \sqrt{\omega_0^2-\gamma^2}$,

$$g(t) = e^{-\gamma t}\,\frac{\sin(\omega t)}{\omega}.$$

In summary,

$$g(t) = \begin{cases} e^{-\gamma t}\,\dfrac{\sinh(\alpha t)}{\alpha}, & \text{two real roots} \\ t e^{-\gamma t}, & \text{double root} \\ e^{-\gamma t}\,\dfrac{\sin(\omega t)}{\omega}, & \text{complex roots} \end{cases}$$

$\alpha = \sqrt{\gamma^2-\omega_0^2}$, $\quad \omega = \sqrt{\omega_0^2-\gamma^2} = i\alpha$

In principle, this reduces the solution of the equation

$$\frac{d^2y}{dt^2} + 2\gamma\frac{dy}{dt} + \omega_0^2 y = F(t)$$

to finding the integral $(F * g)(t)$. In this sense, Green's function characterizes the response of the equation to forcing functions.

Green's functions and transfer functions will be explored further

In later courses on control theory, and signals and systems (especially LTI = Linear, Time-Invariant systems)

The idea of Green's function is broader, and appears in other settings in differential equations, including time-varying equations and partial differential equations.