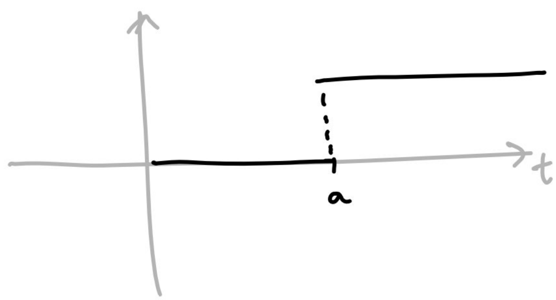
Mthe 237
Lecture 26
Nov. 08, 2017

Topic : Examples of working with piecewise-defined forcing functions

§ Translations and the Laplace transform

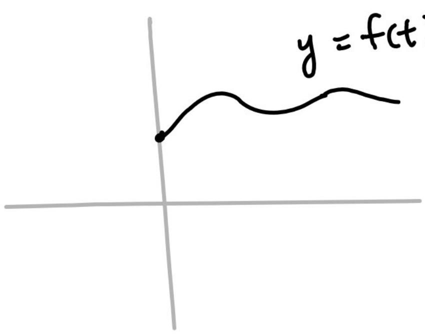
Reminder from last time:

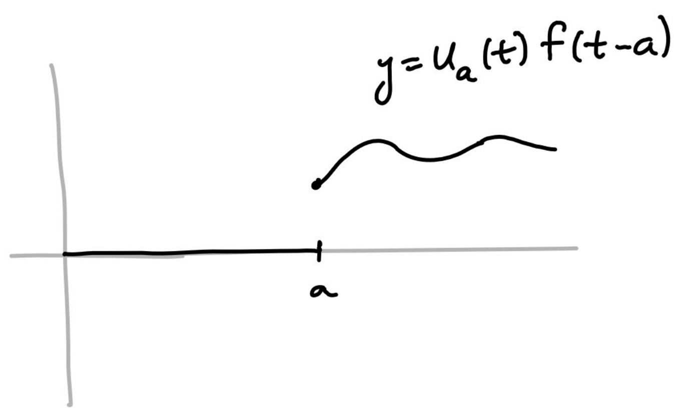
$$u_a(t) = \begin{cases} 1, & t \geq a \\ 0, & 0 \leq t < a \end{cases}$$



is the unit step function with jump at $t=a$

We denote by $u_a(t)\, f(t-a)$ the translate of the function f by a in the positive t direction.





Prop. Suppose $\mathcal{L}[f](s_0)$ converges. Then for any $a \geq 0$, $\mathcal{L}[u_a(t)f(t-a)](s_0)$ also converges and

$$\mathcal{L}[u_a(t)f(t-a)](s_0) = e^{-as}\mathcal{L}[f](s_0).$$

Proof: $\mathcal{L}[u_a(t)f(t-a)](s)$

$$= \int_0^\infty u_a(t)f(t-a)\,e^{-st}\,dt$$

$$= \int_a^\infty f(t-a)\,e^{-st}\,dt.$$

Use the change of variable $u = t-a$, $du = dt$

$$= \int_0^\infty f(u)\,e^{-s(u+a)}\,du$$

$$= e^{-as}\int_0^\infty f(u)\,e^{-su}\,du.$$

$$= e^{-as}\mathcal{L}[f](s).$$

Corollary: If $f = \mathcal{L}^{-1}[F]$, then

$$u_a(t)f(t-a) = \mathcal{L}^{-1}[e^{-as}F].$$

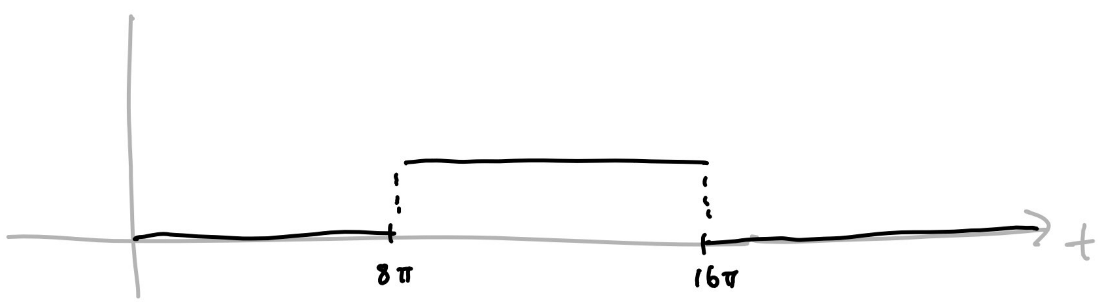
This proposition lets us work cleanly with forcing functions that are defined piecewise. For the rest of today, we work out two examples for the differential equation

$$\frac{d^2y}{dt^2} + \frac{1}{2}\frac{dy}{dt} + \frac{5}{16}y = F(t), \qquad y(0)=0, \quad \frac{dy}{dt}(0)=0,$$

which can be thought of as an underdamped harmonic oscillator starting from rest.

§ Examples

1. $F(t) = u_{8\pi}(t) - u_{16\pi}(t)$



Applying the Laplace transform to both sides of our differential equation, we get

$$s^2\mathcal{L}[y](s) + \frac{s}{2}\mathcal{L}[y](s) + \frac{5}{16}\mathcal{L}[y](s) = \frac{e^{-8\pi s}}{s} - \frac{e^{-16\pi s}}{s}$$

$$\left(s^2+\frac{s}{2}+\frac{5}{16}\right)\mathcal{L}[y](s) = \frac{e^{-8\pi s}}{s} - \frac{e^{-16\pi s}}{s}$$

$$\mathcal{L}[y](s) = \left(\frac{1}{(s^2+\frac{s}{2}+\frac{5}{16})s}\right)\left(e^{-8\pi s} - e^{-16\pi s}\right)$$

Let $\quad H(s) = \left(\frac{1}{(s^2+\frac{s}{2}+\frac{5}{16})s}\right)$

and $h(t) = \mathcal{L}^{-1}[H]$.

By the Corollary above, the solution is then

$$y(t) = u_{8\pi}(t)\, h(t-8\pi) - u_{16\pi}(t)\, h(t-16\pi).$$

So it is enough to find h.

First, look for a partial fraction decomposition:

$$\frac{1}{(s^2+\frac{s}{2}+\frac{5}{16})s} = \frac{As+B}{s^2+\frac{s}{2}+\frac{5}{16}} + \frac{C}{s}$$

$$= \frac{As^2+Bs + C(s^2+\frac{s}{2}+\frac{5}{16})}{(s^2+\frac{s}{2}+\frac{5}{16})s}$$

$$\left.\begin{aligned} A+C &= 0 \\ B+\frac{C}{2} &= 0 \\ \frac{5}{16}C &= 1 \end{aligned}\right\} \Rightarrow A = -\frac{16}{5}, \quad B = -\frac{8}{5}, \quad C = \frac{16}{5}$$

$$H(s) = \frac{8}{5}\left(\frac{-2s-1}{s^2+\frac{s}{2}+\frac{5}{16}} + \frac{2}{s}\right)$$

Now recall that

$$\mathcal{L}[e^{\sigma t}\cos(\omega t)](s) = \frac{(s-\sigma)}{(s-\sigma)^2+\omega^2},$$

$$\mathcal{L}[e^{\sigma t}\sin(\omega t)](s) = \frac{\omega}{(s-\sigma)^2+\omega^2}.$$

$\frac{2s+1}{s^2+\frac{s}{2}+\frac{5}{16}}$ can be written as a linear combination of these two transforms.

Completing the square,

$$s^2+\frac{s}{2}+\frac{5}{16} = \left(s+\frac{1}{4}\right)^2 - \frac{1}{16} + \frac{5}{16}$$

$$= \left(s+\frac{1}{4}\right)^2 + \frac{1}{4}.$$

So $\sigma = -\frac{1}{4}$, $\omega = \frac{1}{2}$.

$2s+1 = 2\left(s+\frac{1}{4}\right) + \frac{1}{2}$, so

$$\frac{2s+1}{s^2+\frac{s}{2}+\frac{5}{16}} = 2\,\frac{s+\frac{1}{4}}{(s+\frac{1}{4})^2+\frac{1}{4}} + \frac{\frac{1}{2}}{(s+\frac{1}{4})^2+\frac{1}{4}}$$

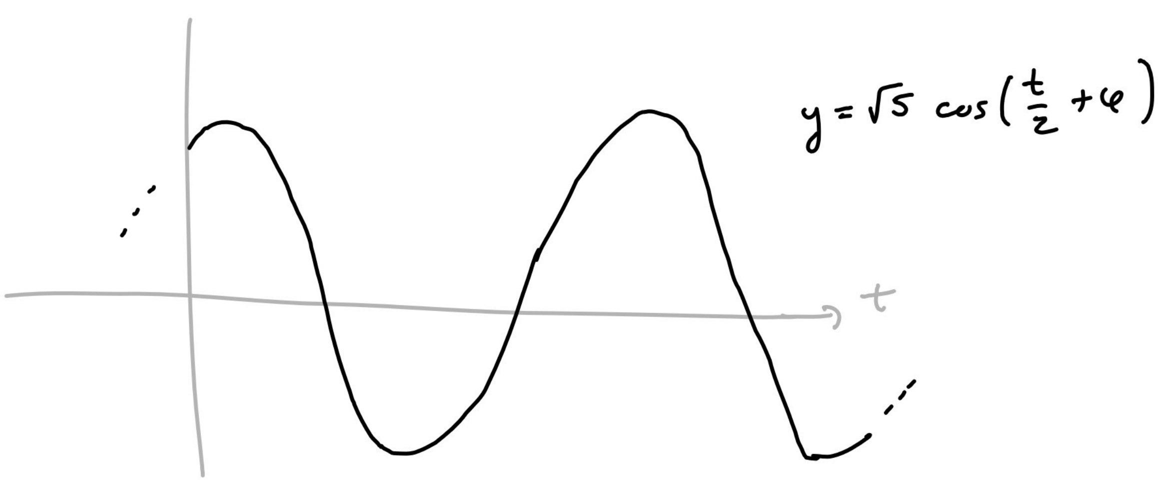
$$\mathcal{L}^{-1}\left[\frac{2s+1}{s^2+\frac{s}{2}+\frac{5}{16}}\right] = 2e^{-t/4}\cos\left(\frac{t}{2}\right) + e^{-t/4}\sin\left(\frac{t}{2}\right).$$

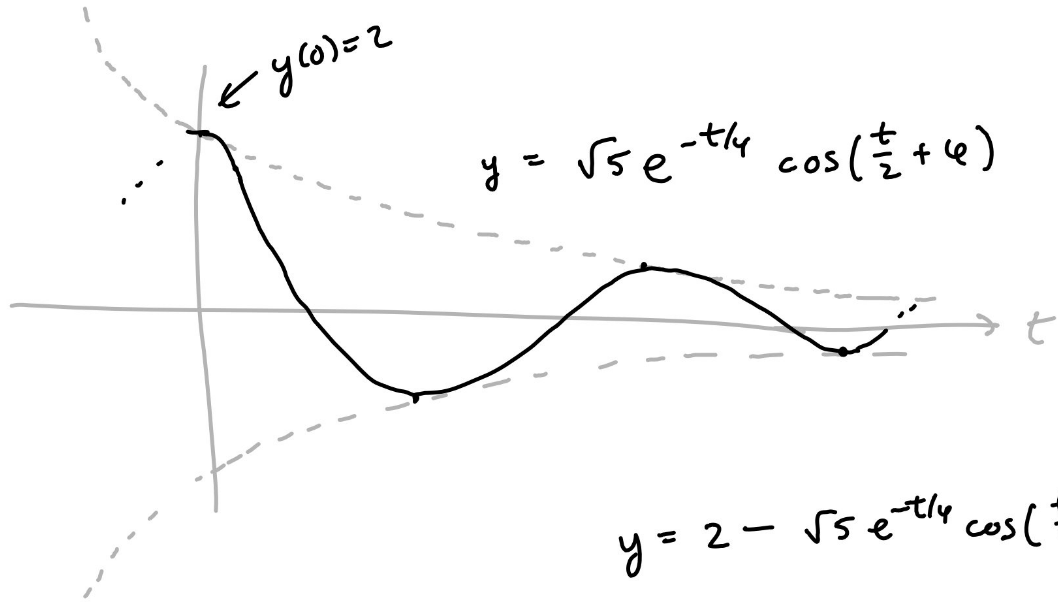
So that finally

$$h(t) = \mathcal{L}^{-1}[H] = \frac{8}{5}\left(2 - 2e^{-t/4}\cos\left(\frac{t}{2}\right) - e^{-t/4}\sin\left(\frac{t}{2}\right)\right)$$

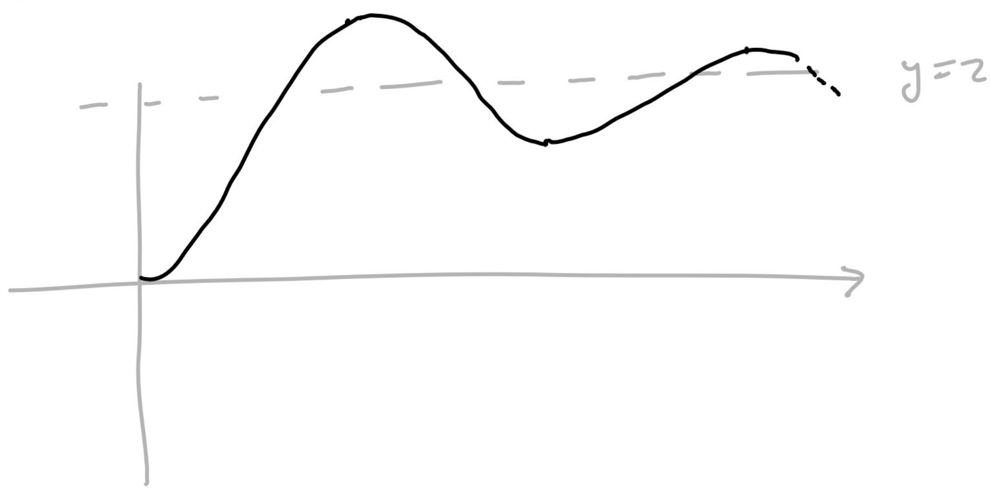
To understand the graph of h, it is helpful to switch the last two terms to phase-amplitude form

$$\left[\begin{array}{l} a_1\cos(\omega t) + a_2\sin(\omega t) = A\cos(\omega t + \varphi), \\ \quad \text{where} \\ \qquad A = \sqrt{a_1^2 + a_2^2} \\ \qquad \tan\varphi = -\frac{a_2}{a_1} \\ \text{So} \quad e^{-t/4}\left(2\cos\left(\frac{t}{2}\right) + \sin\left(\frac{t}{2}\right)\right) \\ \quad = e^{-t/4}\sqrt{5}\cos\left(\frac{t}{2} + \varphi\right), \quad \text{where } \varphi = \arctan\left(-\frac{1}{2}\right) \approx -26^\circ \end{array}\right]$$





$$y = 2 - \sqrt{5}\, e^{-t/4} \cos\left(\frac{t}{2} + \varphi\right)$$

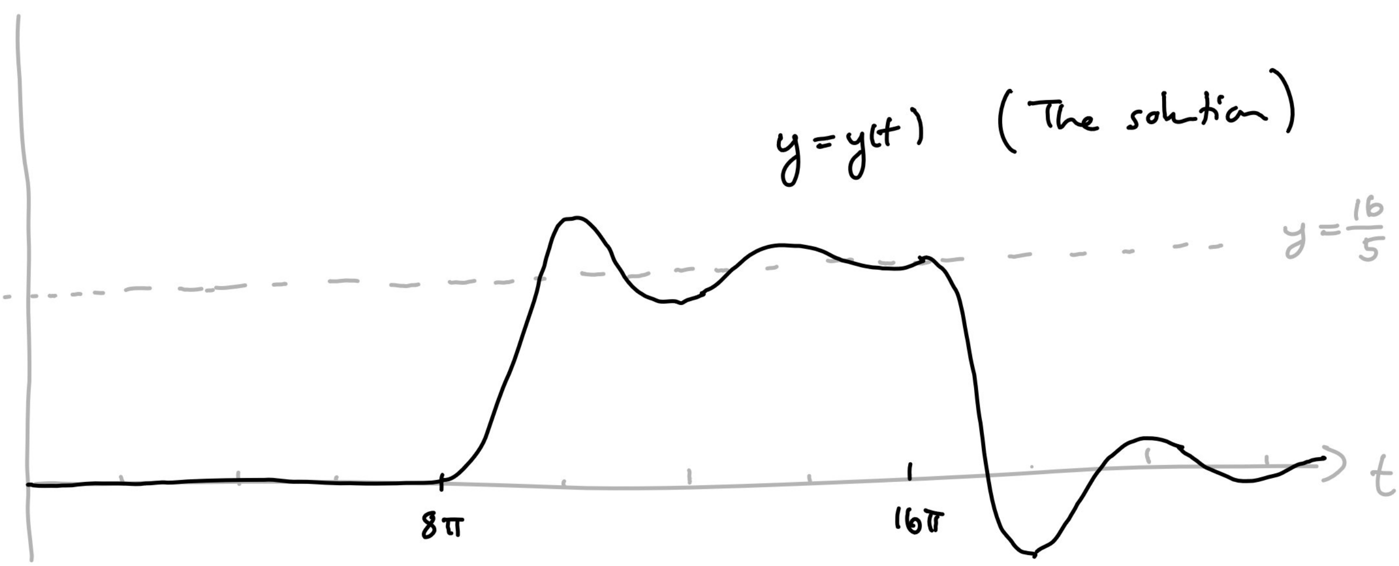


We can think of this graph (up to scaling by $\frac{8}{5}$) as the "system response" to a forcing term that is a unit step function.

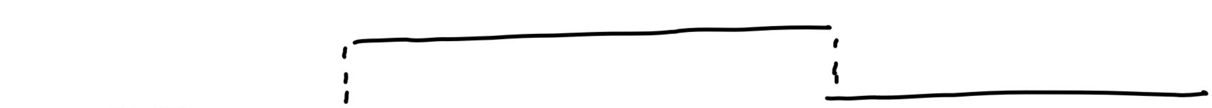
The solution $y(t)$ is a superposition of two such responses, which we can sketch as follows:

$$y(t) = u_{8\pi}(t)\, h(t-8\pi) + u_{16\pi}(t)\, h(t-16\pi)$$

$$= \begin{cases} 0, & 0 \le t < 8\pi \\ \frac{8}{5}\left(2 - \sqrt{5}\, e^{-\frac{(t-8\pi)}{4}} \cos\left(\frac{t-8\pi}{2} + \varphi\right)\right) & \\ \quad = \frac{8}{5}\left(2 - \sqrt{5}\, e^{-(t-8\pi)/4} \cos\left(\frac{t}{2} + \varphi\right)\right), & 8\pi \le t < 16\pi \\ \frac{8}{5}\left(2 - \sqrt{5}\, e^{-(t-8\pi)/4} \cos\left(\frac{t}{2} + \varphi\right)\right) & \\ \quad - \frac{8}{5}\left(2 - \sqrt{5}\, e^{-(t-16\pi)/4} \cos\left(\frac{t-16\pi}{2} + \varphi\right)\right) & \\ \quad = \frac{8}{5}\left(\sqrt{5}\left(e^{-(t-16\pi)/4} - e^{-(t-8\pi)/4}\right) \cos\left(\frac{t}{2} + \varphi\right)\right), & t > 16\pi \end{cases}$$

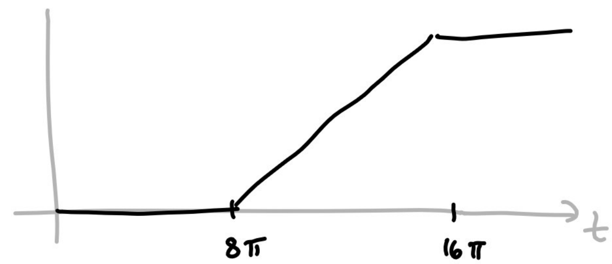


F(t):



2. Consider the same system with an input that ramps between two constant values:

$$F(t) = \begin{cases} 0, & 0 < t < 8\pi \\ t - 8\pi, & 8\pi \le t < 16\pi \\ 8\pi, & t \geq 16\pi \end{cases}$$



$$F(t) = u_{8\pi}(t)(t-8\pi) - u_{16\pi}(t)(t-16\pi)$$

$$\mathcal{L}[F](s) = \frac{e^{-8\pi s}}{s^2} - \frac{e^{-16\pi s}}{s^2}$$

Proceeding similarly to the previous example, we have

$$\mathcal{L}[y](s) = \underbrace{\left(\frac{1}{(s^2 + \frac{s}{2} + \frac{5}{16})s^2}\right)}_{H(s)}\left(e^{-8\pi s} - e^{-16\pi s}\right)$$

Let $h(t) = \mathcal{L}^{-1}[H]$

Then $y(t) = u_{8\pi}(t)\,h(t-8\pi) - u_{16\pi}(t)\,h(t-16\pi).$

We decompose $H(s)$ into partial fractions:

$$\frac{1}{(s^2+\frac{s}{2}+\frac{5}{16})s^2} = \frac{As+B}{s^2+\frac{s}{2}+\frac{5}{16}} + \frac{C}{s} + \frac{D}{s^2}$$

$$= \frac{As^3+Bs^2+C(s^3+\frac{s^2}{2}+\frac{5}{16}s)+D(s^2+\frac{s}{2}+\frac{5}{16})}{(s^2+\frac{s}{2}+\frac{5}{16})s^2}$$

$$\left.\begin{aligned} A + C &= 0 \\ B+\frac{C}{2}+D &= 0 \\ \frac{5}{16}C+\frac{D}{2} &= 0 \\ \frac{5}{16}D &= 1 \end{aligned}\right\} \qquad \begin{aligned} &D=\frac{16}{5}, \quad C = -\frac{8}{5}\cdot\frac{16}{5} \\ &A = \frac{8}{5}\cdot\frac{16}{5} \\ &B = \frac{16}{5}\left(\frac{4}{5}-1\right) = -\frac{16}{25} \end{aligned}$$

$$H(s) = \frac{16}{25}\left(\frac{8s-1}{s^2+\frac{s}{2}+\frac{5}{16}} - \frac{8}{s} + \frac{5}{s^2}\right)$$

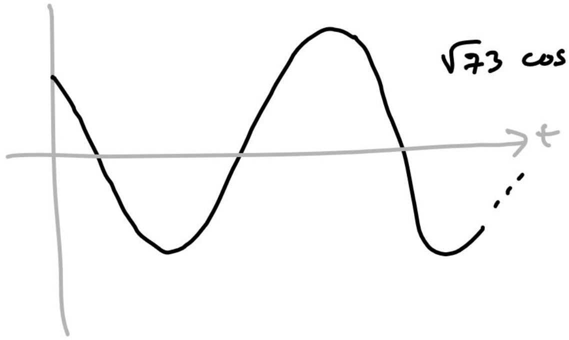
Now,

$$8s-1 = 8(s+\tfrac{1}{4})-3,$$ so

$$h(t) = \mathcal{L}^{-1}[H] = \frac{16}{25}\left(8e^{-t/4}\cos(\tfrac{t}{2}) - 3e^{t/4}\sin(\tfrac{t}{2}) - 8 + 5t\right)$$

$$= \frac{16}{25}\left(\sqrt{73}\,e^{-t/4}\cos(\tfrac{t}{2}+\varphi) - 8 + 5t\right)$$

$$\varphi = \arctan(\tfrac{3}{8}) \approx 21^\circ$$



$\sqrt{73}\cos\left(\frac{t}{2}+\varphi\right)$

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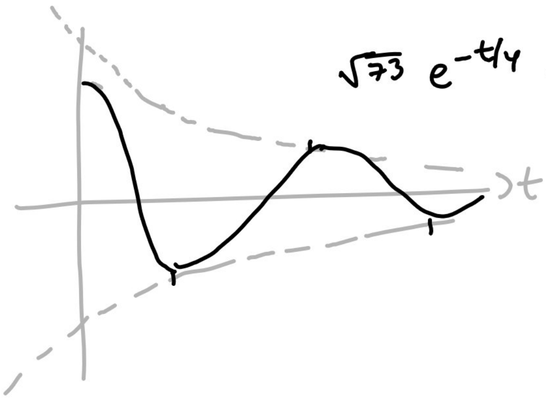
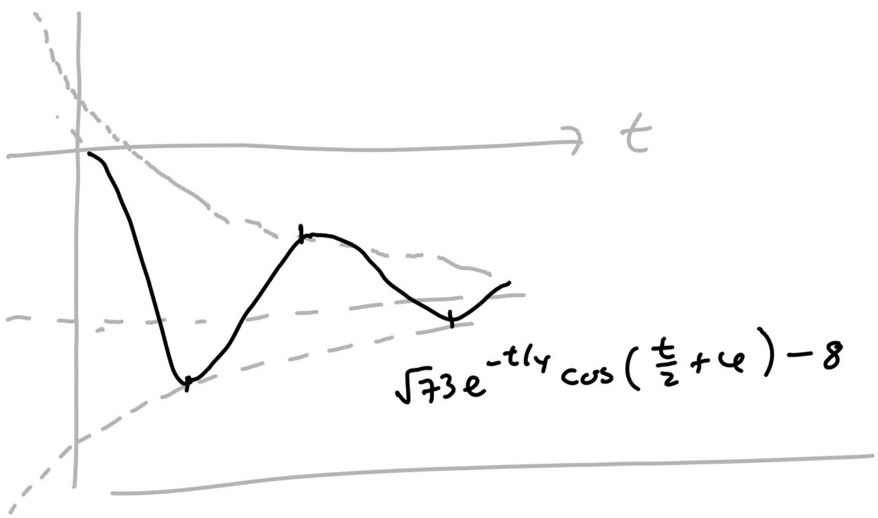
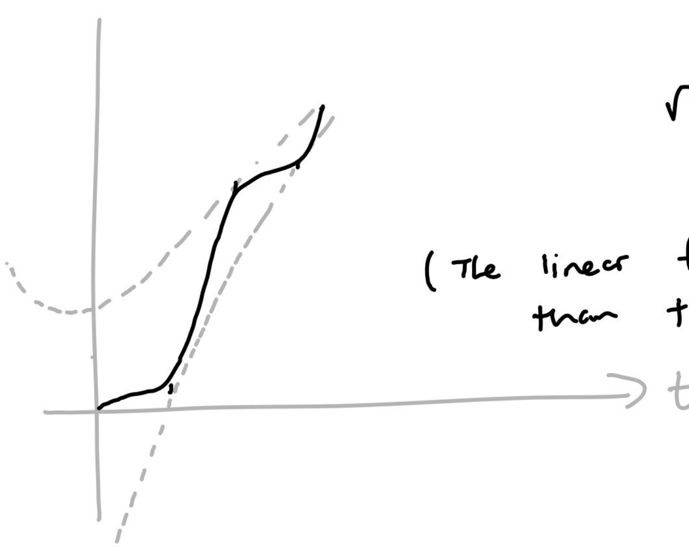


$\sqrt{73}\, e^{-t/4}\cos\left(\frac{t}{2}+\varphi\right)$



$\sqrt{73}\, e^{-t/4}\cos\left(\frac{t}{2}+\varphi\right)-8$

$\sqrt{73}\, e^{-t/4}\cos\left(\frac{t}{2}+\varphi\right)-8+5t$



(The linear function grows faster than the cosine dips)

$$y(t) = u_{8\pi}(t)\,h(t-8\pi) - u_{16\pi}(t)\,h(t-16\pi)$$

$$= 0 \quad \text{for} \quad 0 \le t < 8\pi,$$

$$\frac{16}{25}\left(\sqrt{73}\, e^{-(t-8\pi)/4} \cos\left(\frac{t}{2} + \varphi\right) - 8 + 5(t-8\pi)\right), \quad 8\pi \le t < 16\pi$$

$$\frac{16}{25}\left(\sqrt{73}\left(e^{-(t-8\pi)/4} - e^{-(t-16\pi)/4}\right)\cos\left(\frac{t}{2} + \varphi\right) + 40\pi\right), \quad t > 16\pi$$

Sketch of the solution: