Mthe 237
Lecture 25
Nov. 07, 2017

Topics: • Completing the list of Laplace transforms of quasipolynomials
• Using Step Functions

§ Taking a derivative inside an integral

Suppose that we are investigating an integral that depends on a parameter s, for instance

$$\int_0^1 (t+s^3)^2 \, dt$$

Computing the integral, we have

$$\frac{1}{3}(t+s^3)^3 \Big|_{t=0}^{t=1}$$

$$= \frac{1}{3}\left((1+s^3)^3 - s^9\right)$$

$$= \frac{1}{3}\left(1+3s^3+3s^6+s^9-s^9\right)$$

$$= \frac{1}{3}+s^3+s^6.$$

Treating this as a function of s, we can find its derivative –

$$\frac{d}{ds}\int_0^1 (t+s^3)^2 \, dt = 3s^2+6s^5.$$

Let's compare the result to the one we would have obtained if instead we took the derivative "inside the integral":

$$\int_0^1 \frac{\partial}{\partial s} (t+s^3)^2 \, dt$$

$$= \int_0^1 2(t+s^3) \cdot 3s^2 \, dt$$

$$= 3s^2 (t+s^3)^2 \Big|_{t=0}^{t=1}$$

$$= 3s^2 \left[(1+s^3)^2 - s^6 \right]$$

$$= 3s^2 \left[1 + 2s^3 + s^6 - s^6 \right]$$

$= 3s^2 + 6s^5.$ The results are equal!

We have $\frac{d}{ds} \int_0^1 (t+s)^2 \, dt = \int_0^1 \frac{\partial}{\partial s} (t+s)^2 \, dt.$

In general, it is not true that a derivative can be taken inside an integral in this way. It is useful to have criteria for when it is possible. One such criterion is —

<u>Theorem.</u> Suppose that $f(t,s)$ and $\frac{\partial}{\partial s} f(t,s)$ are continuous functions in the region

$$\{(t,s) \in \mathbb{R}^2 : \ a \le t \le b, \quad s_0 - c \le s \le s_0 + c\}$$ (where $c > 0$ is real number)

Suppose that

$$|f(t,s)| \le A(t) \quad \text{and} \quad \left| \frac{\partial f}{\partial s}(t,s) \right| \le B(t)$$

where

$$\int_a^b A(t)\, dt \quad \text{and} \quad \int_a^b B(t)\, dt \quad \text{converge.}$$

(Integrals could be indefinite)

Then

$$\frac{d}{ds} \int_a^b f(t,s)\, dt = \int_a^b \frac{\partial}{\partial s} f(t,s)\, dt.$$

Holds for $s = s_0$

<u>Reference:</u> Lang, Undergraduate Analysis, pp. 337-339.

Suppose that $f(t)$ is of exponential order a, so that $|f(t)| \le M e^{at}$ for $t \ge t_0 \ge 0$

For any $s > \sigma > a$

$$|f(t) e^{-st}| \le M e^{(a-s)t} \le M e^{(a-\sigma)t} \quad \text{for } t \ge t_0$$

and

$$\frac{\partial}{\partial s}\left(f(t) e^{-st}\right) = -s f(t) e^{-st}, \quad \text{so}$$

$$\left| \frac{\partial}{\partial s}(f(t) e^{-st}) \right| \le s M e^{(a-\sigma)t} \le (s_0 + c) M e^{(a-\sigma)t}$$

We can take $A(t) = Me^{(a-\sigma)t}$ and

$$B(t) = (s_0 + c)\, Me^{(a-\sigma)t}$$

and apply the Theorem to conclude:

Prop. Suppose that f is of exponential order a, and is continuous. For all $s > a$,

$\mathcal{L}[tf](s)$ exists and

$$\mathcal{L}[tf](s) = -\frac{d}{ds}\mathcal{L}[f](s)$$

Proof:

$$\frac{d}{ds}\mathcal{L}[f](s) = \frac{d}{ds}\int_0^\infty f(t)e^{-st}\,dt$$

$$= \int_0^\infty \frac{\partial}{\partial s} f(t)e^{-st}\,dt$$

$$= \int_0^\infty -t f(t) e^{-st}\,dt$$

$$= -\int_0^\infty t f(t) e^{-st}\,dt$$

$$= -\mathcal{L}[tf](s).$$

Multiplying both sides by (-1) shows the claim. ∎

Corollary Same hypotheses on f, for all $s > a$, $k \in \mathbb{N}$

$$\mathcal{L}[t^k f](s) = (-1)^k \frac{d^k}{ds^k}\mathcal{L}[f](s).$$

§ Laplace transforms of Quasipolynomials

We have computed the following transforms in class and in tutorial:

$$\mathcal{L}[e^{\sigma t}](s) = \frac{1}{s-\sigma}, \qquad s > \sigma$$

$$\mathcal{L}[1](s) = \frac{1}{s}, \qquad s > 0$$

$$\mathcal{L}[\cos(\omega t)](s) = \frac{s}{s^2+\omega^2}, \qquad s > 0$$

$$\mathcal{L}[\sin(\omega t)](s) = \frac{\omega}{s^2+\omega^2}, \qquad s > 0$$

Using the proposition just proved we can, at least in principle, complete this list to include all quasipolynomials.

With this done, the Laplace transform method can be applied to solve any linear equation that can be done using the method of undetermined coefficients (again, at least in principle).

$$\mathcal{L}[te^{\sigma t}](s) = -\frac{d}{ds}\left(\frac{1}{s-\sigma}\right)$$

$$= -\left(-\frac{1}{(s-\sigma)^2}\right)$$

$$= \frac{1}{(s-\sigma)^2}$$

$$\mathcal{L}[t^2e^{\sigma t}](s) = (-1)^2\frac{d^2}{ds^2}\left(\frac{1}{s-\sigma}\right)$$

$$= \frac{2}{(s-\sigma)^3}$$

Similarly (as can be checked by induction), for $k \in \mathbb{N}$,

$$\mathcal{L}[t^k e^{\sigma t}](s) = \frac{1\cdot 2\cdot 3\cdot 4\cdots k}{(s-\sigma)^{k+1}} = \frac{k!}{(s-\sigma)^{k+1}}$$

As a special case when $\sigma = 0$, we get

$$\mathcal{L}[t^k](s) = \frac{k!}{s^{k+1}} \quad \text{for any } k \in \mathbb{N}.$$

$$\mathcal{L}\left[t\cos(\omega t)\right](s) = -\frac{d}{ds}\ \frac{s}{s^2+\omega^2}$$

Quotient rule: $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$

$$-\frac{d}{ds}\frac{s}{s^2+w^2} = -\frac{1\cdot(s^2+w^2)-s(2s)}{(s^2+w^2)^2}$$

$$= -\frac{w^2-s^2}{(s^2+w^2)^2}$$

$$= \frac{s^2-w^2}{(s^2+w^2)^2}$$

$$\mathcal{L}[t\sin(wt)](s) = -\frac{d}{ds}\frac{w}{s^2+w^2}$$

$$= \frac{w}{(s^2+w^2)^2}\cdot 2s$$

$$= \frac{2ws}{(s^2+w^2)^2}.$$

In principle, we can find

$$\mathcal{L}[t^k\cos(wt)](s) = (-1)^k\frac{d^k}{ds^k}\mathcal{L}[\cos(wt)](s) \text{ and}$$

$$\mathcal{L}[t^k\sin(wt)](s) = (-1)^k\frac{d^k}{ds^k}\mathcal{L}[\sin(wt)](s).$$

There are closed-form expressions for both, which are quite messy. It is worth observing that

$$\mathcal{L}[t^k\cos(wt)] = \frac{\text{Polynomial in } s,w}{(s^2+w^2)^{k+1}}$$

(also true for $\mathcal{L}[t^k\sin(wt)]$)

Finally, you'll show on the homework that

$$\mathcal{L}[e^{\sigma t}\cos(\omega t)](s) = \frac{(s-\sigma)}{(s-\sigma)^2+\omega^2} \quad \text{and}$$

$$\mathcal{L}[e^{\sigma t}\sin(\omega t)](s) = \frac{\omega}{(s-\sigma)^2+\omega^2}.$$

Again, expressions for $\mathcal{L}[t^k e^{\sigma t}\cos(\omega t)]$ and $\mathcal{L}[t^k e^{\sigma t}\sin(\omega t)]$ can be found by differentiation.

Much more extensive tables of Laplace transforms have been computed, but for this course we'll stop here.

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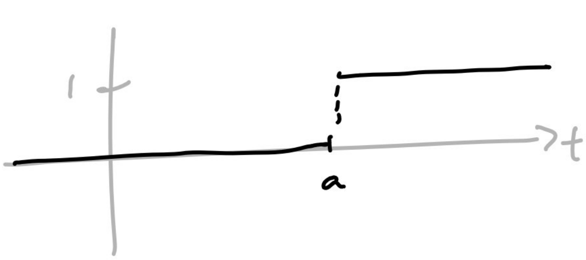
§ Using Step Functions

One of the advantages of the Laplace transform method is that it allows us to cleanly work with forcing functions $F(t)$ that are defined piecewise.

For this, it is useful to define

$$u_a(t) = \begin{cases} 1, & t \geq a \\ 0, & \text{otherwise} \end{cases}$$

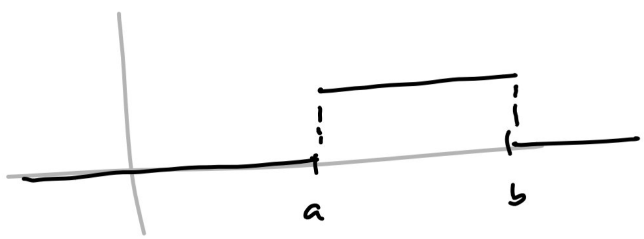


This is called a unit step function at time a (or Heaviside step function)

Examples of functions defined using step functions:

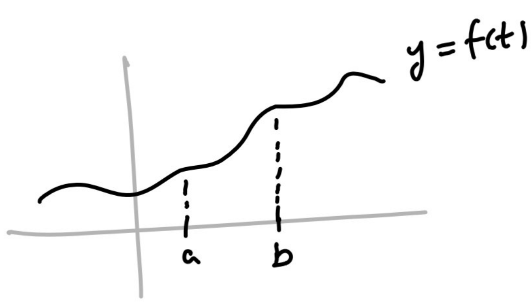
For $b > a$

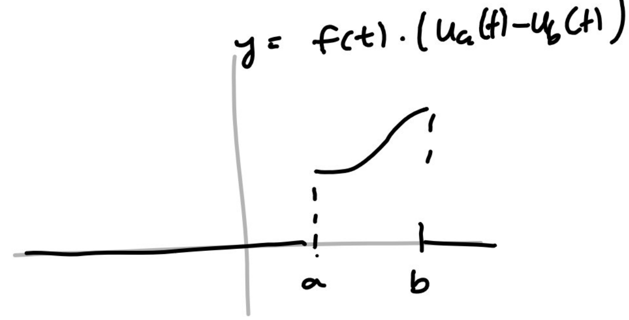
$$u_a(t) - u_b(t) = \begin{cases} 1, & a \le t \le b \\ 0, & \text{otherwise} \end{cases}$$



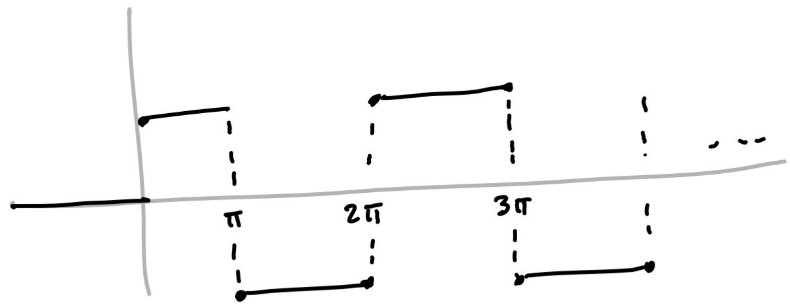
For any $f(t)$,

$\big(u_a(t) - u_b(t)\big) \cdot f(t)$ "filters out" the graph of f outside of $a \le t \le b$.





Describing a square wave:

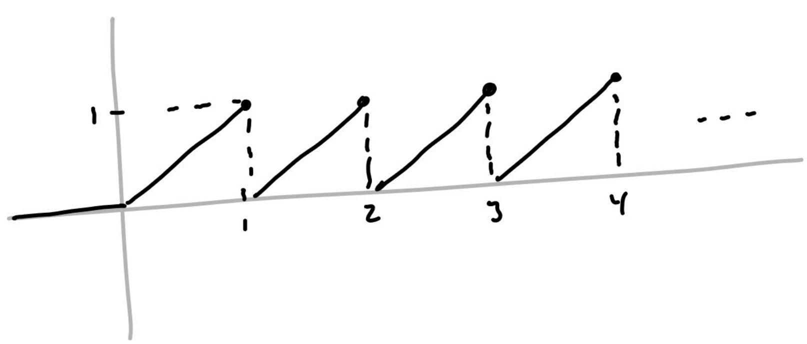


$$\left(u_0(t) - u_\pi(t)\right) - \left(u_\pi(t) - u_{2\pi}(t)\right) + \left(u_{2\pi}(t) - u_{3\pi}(t)\right) - \cdots$$

$$= u_0(t) - 2u_\pi(t) + 2u_{2\pi}(t) - 2u_{3\pi}(t) + \cdots$$

$$= u_0(t) + 2\sum_{n=1}^{\infty} (-1)^n u_{n\pi}(t)$$

Describing a sawtooth wave:



$$\left(u_0(t) - u_1(t)\right)\cdot t + \left(u_1(t) - u_2(t)\right)(t-1) + \cdots$$

$$\cdots + \left(u_n(t) - u_{n+1}(t)\right)(t-n) + \cdots$$

$$= \sum_{n=0}^{\infty} \left(u_n(t) - u_{n+1}(t)\right)(t-n)$$

Laplace transform of a step function:

For $a \geq 0$,

$\mathcal{L}[u_a(t)](s)$

$$= \int_0^\infty u_a(t)\, e^{-st}\, dt$$

$$= \int_a^\infty e^{-st}\, dt \,. \qquad \text{Let } \begin{array}{l} v = t - a \\ dv = dt \end{array}$$

$$= \int_0^\infty e^{-s(a+v)}\, dv$$

$$= e^{-as} \int_0^\infty e^{-sv}\, dv$$

$$= e^{-as}\, \mathcal{L}[1](s)$$

$$= \frac{e^{-as}}{s}, \quad s > 0$$

Thus,

- $\mathcal{L}\left[u_a(t) - u_b(t)\right](s) = \dfrac{e^{-as}}{s} - \dfrac{e^{-bs}}{s}$,
- $\mathcal{L}\left[u_0(t) + 2\sum_{n=1}^{\infty} (-1)^n u_{n\pi}(t)\right](s)$

$$= \frac{1}{s} + 2\sum_{n=1}^{\infty} (-1)^n \frac{e^{-n\pi s}}{s}, \quad \text{and so on...}$$