MTHE 237
Lecture 24
Nov. 03, 2017

Topic: Some justification of the Laplace transform method
(Not for examination – read if interested!)

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As part of the method of solving linear differential equations using the Laplace transform that we have seen last time, it is necessary to make use of the result

$$(*) \quad \mathcal{L}\left[\frac{d^r y}{dt^r}\right](s) = s^r \mathcal{L}[f](s) - s^{r-1} f(0) - \cdots - \frac{d^{r-1} f}{dt^{r-1}}(0)$$

relating the Laplace transform of $\frac{d^r y}{dt^r}$ with that of $y$.

However, our proof of (*) required some hypotheses: namely that $y, \frac{dy}{dt}, \ldots, \frac{d^{r-1}y}{dt^{r-1}}$ are functions of <u>exponential order</u>. How can we check this without first solving the differential equation?

The goal of today is to show that if the forcing term $F(t)$ of a linear eq. (with constant coeffs.) is of exponential order, so are the solution $\varphi$ and its derivatives, thereby justifying this step of the method (at least as long as the forcing term is of exponential order).

Along the way, we'll prove that quasipolynomials have exponential order.

Reminder: A function $f$ is said to be of exponential order $a$ if there exist $M>0$, $t_0 \in \mathbb{R}$ so that

$$|f(t)| \le Me^{at}, \quad \text{for all } t \ge t_0.$$

**Def.** Say a function $f$ is of exponential order if it is of exponential order $a$ for some $a \in \mathbb{R}$.

**Lemma.** Suppose that $f:[0,\infty) \to \mathbb{R}$ is of exponential order $a$

$g:[0,\infty) \to \mathbb{R}$ is of exponential order $b$

Then

i) $f+g$ is of exponential order $\max(a,b)$

ii) $fg$ is of exponential order $a+b$

iii) $cf$ is of exponential order $a$ ($c \in \mathbb{R}$).

**Proof:** Suppose $|f(t)| \le M_1 e^{at}$ for $t \ge t_1$, $|g(t)| \le M_2 e^{bt}$ for $t \ge t_2$

i) $|f(t)+g(t)| \le |f(t)| + |g(t)|$

$\le M_1 e^{at} + M_2 e^{bt}$, $\quad t \ge \max(t_1,t_2)$

Triangle inequality

$\le (2\max(M_1,M_2))\, e^{\max(a,b)t}$, $\quad t \ge \max(t_1,t_2)$

So $f(t)+g(t)$ is of exponential order $\max(a,b)$.

ii) $|f(t)g(t)| = |f(t)|\,|g(t)|$

$$\leq (M_1 e^{at})(M_2 e^{bt}), \quad t \geq \max(t_1, t_2)$$

$$= (M_1 M_2)\, e^{(a+b)t}, \quad t \geq \max(t_1, t_2)$$

iii) $|cf(t)| = |c|\,|f(t)|$

$$\leq (|c| M_1)\, |f(t)|, \quad t \geq t_1$$

(Alternatively, notice that constant functions are of exponential order 0 and apply ii))

$\blacksquare$

Some terminology: These three properties show that the set of all functions of exponential order has the structure of an "$\mathbb{R}$-algebra" (roughly, a real vector space, together with a compatible multiplication).

<u>Reminder</u>: A quasipolynomial is a solution of a linear homogeneous differential equation with constant coefficients.

More concretely, it is a function of the form

$e^{\sigma t} \cdot$ Polynomial in $t$

$e^{\sigma t} \cos(\omega t) \cdot$ Poly in $t$

$e^{\sigma t} \sin(\omega t) \cdot$ Poly in $t$, or a sum of such functions.

**Prop.** Every quasipolynomial is a function of exponential order.

**Proof.** By iterating the results of the lemma, we see that it is sufficient to show that the functions

$$e^{\sigma t}, \quad \cos(\omega t), \quad \sin(\omega t), \quad t$$

have exponential order.

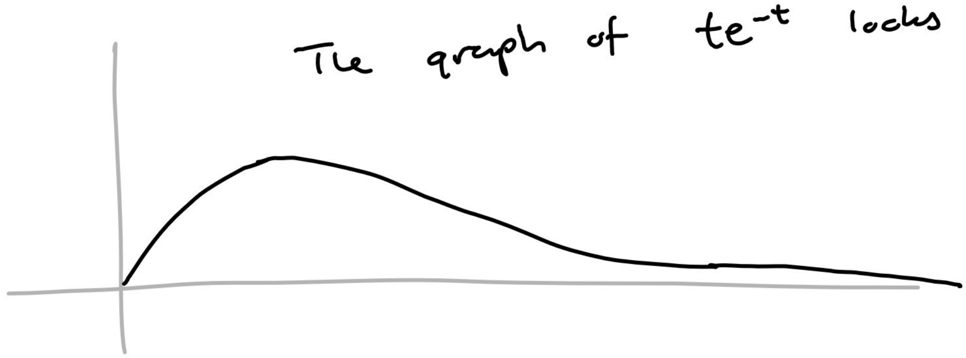
- $|\cos(\omega t)| \leq 1$, $|\sin(\omega t)| \leq 1$ so $\cos(\omega t)$, $\sin(\omega t)$ have exponential order 0.

- $|e^{\sigma t}| = e^{\sigma t}$. So $e^{\sigma t}$ has exponential order $\sigma$.

- By l'Hopital's rule, $\displaystyle\lim_{t\to\infty} \frac{t}{e^t} = \lim_{t\to\infty} \frac{1}{e^t} = 0$

  Also, $te^{-t}\big|_{t=0} = 0$.

  The graph of $te^{-t}$ looks like



$(te^{-t})' = (1-t)e^{-t}$ (Product rule)

$(te^{-t})'' = -e^{-t} - (1-t)e^{-t} = (t-2)e^{-t}$

$(te^{t})'$ has a single zero at $t=1$.

$(te^{-t})''\big|_{t=1} = -e^{-1} < 0$, so this is a max

$te^{-t}\big|_{t=1} = e^{-1}$.

This shows that $te^{-t} \leq e^{-1}$ for all $t \geq 0$, or

$$t \leq e^{-1} e^{t},$$

so that the function $t$ is of exponential order 1.

(In fact, this is not optimal, but sufficient for our purposes here.)

$\blacksquare$

We need one final proposition before coming to the promised proof.

**Prop.** Suppose that $f'$ is of exponential order. Then so is $f$.

Proof: Suppose $|f'(t)| \le Me^{at}$ for all $t \ge t_0$, $a > 0$.

By the fundamental theorem of calculus,

$$f(t) = \int_{t_0}^{t} f'(s)\, ds + f(t_0).$$

Therefore,

$$
\begin{aligned}
|f(t)| &\le \left| \int_{t_0}^{t} f'(s)\, ds \right| + |f(t_0)| \\
&\le \int_{t_0}^{t} |f'(s)|\, ds + |f(t_0)| \\
&\le \int_{t_0}^{t} Me^{as}\, ds + |f(t_0)| \\
&= \frac{M}{a}\left(e^{at} - e^{at_0}\right) + |f(t_0)| \\
&\le \frac{M}{a} e^{at} + |f(t_0)| \\
&\le (\text{constant}) \cdot e^{at} \qquad \text{for } t \ge t_0
\end{aligned}
$$

Triangle inequality

Triangle inequality for $\int$

$$\left( |f(t_0)| = \frac{|f(t_0)|}{e^{at_0}} \cdot e^{at_0} \le \frac{|f(t_0)|}{e^{at_0}} \cdot e^{at} \text{ for } t \ge t_0, \right.$$

$$\left. \text{so } \frac{M}{a} e^{at} + |f(t_0)| \le \left( \frac{M}{a} + \frac{|f(t_0)|}{e^{at_0}} \right) e^{at} \right)$$

This detail is not very important

$\square$

Consider the differential equation

$$(L) \quad \frac{d^r y}{dt^r} + a_{r-1}(t)\frac{d^{r-1}y}{dt^{r-1}} + \cdots + a_1(t)\frac{dy}{dt} + a_0(t)y = F(t)$$

Let $\varphi_1, \ldots, \varphi_r$ denote a linearly independent collection of solutions to the associated homogeneous equation

$$(H_L) \quad \frac{d^r y}{dt^r} + a_{r-1}(t)\frac{d^{r-1}y}{dt^{r-1}} + \cdots + a_1(t)\frac{dy}{dt} + a_0(t)y = 0.$$

<u>Theorem</u>. If $\varphi_1, \ldots, \varphi_r$ and $F$ are functions of exponential order, and $a_{r-1}(t)$ is constant, then any solution $\varphi$ of $(L)$ also has exponential order.

Moreover, if the derivatives

$$\left.\begin{array}{l} \varphi_1', \ \varphi_1'', \ \ldots, \ \varphi_1^{(r)} \\ \vdots \\ \varphi_r', \ \varphi_r'', \ \ldots, \ \varphi_r^{(r)} \end{array}\right\} \text{ are of exponential order,}$$

then so are $\varphi', \varphi'', \ldots, \varphi^{(r)}$.

<u>Corollary</u>. If $F$ is a function of exponential order, then any solution $\varphi$ of

$$\frac{d^r y}{dt^r} + a_{r-1}\frac{d^{r-1}y}{dt^{r-1}} + \cdots + a_1\frac{dy}{dt} + a_0 y = F \qquad (a_j \in \mathbb{R})$$

is also a function of exponential order, and so are $\varphi', \varphi'', \ldots, \varphi^{(r)}$.

<u>Proof of Corollary</u>: The solutions of the associated homogeneous equation are quasipolynomials, which we have shown have exponential order. Moreover, derivatives of quasipolynomials are again quasipolynomials. Therefore, we can apply the Theorem. ∎

<u>Proof of Theorem</u>:

Construct a solution $\varphi$ using variation of parameters:

Let $u_1, \dots, u_r$ be (differentiable) functions that satisfy the linear system

$$\begin{pmatrix} \varphi_1 & \cdots & \varphi_r \\ \vdots & \ddots & \vdots \\ \varphi_1^{(r-1)} & \cdots & \varphi_r^{(r-1)} \end{pmatrix} \begin{pmatrix} u_1' \\ u_2' \\ \vdots \\ u_r' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ F \end{pmatrix} \quad (*)$$

Every solution of $(L)$ has the form

$$\varphi = \underbrace{u_1\varphi_1 + \dots + u_r\varphi_r}_{\varphi_p} + \underbrace{\sum c_j\varphi_j}_{\varphi_h \text{ - solution of } (H_L)} \quad \text{for some } c_j \in \mathbb{R}$$

We have (because $u_1', \ldots, u_r'$ satisfy $(*)$)

$$\varphi' = u_1 \varphi_1' + \cdots + u_r \varphi_r' + \sum c_j \varphi_j'$$

$$\varphi'' = u_1 \varphi_1'' + \cdots + u_r \varphi_r'' + \sum c_j \varphi_j''$$

$$\vdots$$

$$\varphi^{(r-1)} = u_1 \varphi_1^{(r-1)} + \cdots + u_r \varphi_r^{(r-1)} + \sum c_j \varphi_j^{(r-1)}$$

$$\varphi^{(r)} = u_1 \varphi_1^{(r)} + \cdots + u_r \varphi_r^{(r)} + \sum c_j \varphi_j^{(r)} + F$$

By hypotheses, and the lemma from the beginning, if we can show that $u_1, \ldots, u_r$ are of exponential order, we can conclude that $\varphi, \varphi', \ldots, \varphi^{(r)}$ are of exponential order.

By Cramer's Rule,

$$u_j' = \frac{W_j}{W}.$$

$W_j$ is a polynomial in functions of exponential order (since the determinant is a polynomial in the entries of the matrix), hence is itself of exponential order.

By Abel's Theorem, $W = C\exp\left(-\int a_{r-1}(t)\,dt\right)$

$= C\exp\left(-a_{r-1}t\right)$

(By the hypothesis that $a_{r-1}(t)$ is a constant function)

So $\frac{1}{W} = C\exp\left(a_{r-1}t\right)$, which has exponential order.

Therefore, $u_j' = \frac{W_j}{W}$ is of exponential order.

$\Longrightarrow u_j$ is of exponential order for $j=1,\dots,r$

This shows that $\varphi, \varphi', \dots, \varphi^{(r)}$ are all again of exponential order. $\blacksquare$