MTHE 237
Lecture 23
Nov. 01, 2017

Topic: Laplace Transform method for solving linear equations

Last time: The Laplace transform of an integrable function $f: [0,\infty) \to \mathbb{R}$ is

$$\mathcal{L}[f]: s \longmapsto \int_0^\infty f(t) e^{-st}\, dt,$$

with domain $\{ s \in \mathbb{R} : \int_0^\infty f(t) e^{-st}\, dt \text{ converges} \}$.

It would be useful to find conditions on the function f that guarantee that $\int_0^\infty f(t) e^{-st}\, dt$ converges for some s_0 (then we would also have convergence for all $s > s_0$ by a theorem from last time)

Intuitively, to have convergence at $s = s_0$, the function $f(t)$ shouldn't grow faster than $e^{-s_0 t}$ decays.

Motivated by this intuition, we define

Def. A function f is said to be of exponential order a if there exist $M > 0$, $t_0 \in \mathbb{R}$ so that

$$|f(t)| \leq M e^{at} \quad \text{for all } t \geq t_0$$

Prop. Suppose f is of exponential order a. Then $\lim_{t \to \infty} f(t)e^{-st} = 0$ for all $s > a$.

Proof: We have

$$-|f(t)|e^{-st} \leq f(t)e^{-st} \leq |f(t)|e^{-st} \quad \text{for all } t \geq 0,$$

So it is sufficient to show that

$$\lim_{t \to \infty} |f(t)|e^{-st} = 0.$$

(Then apply "squeeze theorem" from Calculus)

Suppose $|f(t)| \leq Me^{at}$ for $t \geq t_0$. Then

$$0 \leq |f(t)|e^{-st} \leq Me^{at}e^{-st} = Me^{(a-s)t} \quad \text{for } t \geq t_0$$

For $s > a$, $(a-s) < 0$, so

$$Me^{(a-s)t} \to 0 \quad \text{as } t \to \infty.$$

Therefore, $\lim_{t \to \infty} |f(t)|e^{-st} = 0$, which then

proves that

$$\lim_{t \to \infty} f(t)e^{-st} = 0.$$

$\blacksquare$

Here is the promised criterion for convergence:

Prop. Suppose that f is of exponential order a. Then $\int_0^\infty f(t)e^{-st}\,dt$ converges for all $s>$

Proof: This wasn't proved in class. Please see Appendix to this lecture if interested.

With these preliminaries out of the way, we can prove the following result, which is key for the Laplace transform method for solving ODEs:

Theorem. Suppose that f is of exponential order a and differentiable for $t \in [a, \infty)$. Then

$$\mathcal{L}\left[\frac{df}{dt}\right] \quad \text{converges for all } s>a$$

and

$$\mathcal{L}\left[\frac{df}{dt}\right](s) = s\,\mathcal{L}[f](s) - f(0) \quad \text{for all } s>a$$

Integration by parts:

$$\int u'v = uv - \int uv'$$

<u>Proof</u>:

$$\int_0^h \underbrace{f'(t)}_{u'}\ \underbrace{e^{-st}}_{v}\, dt$$

$$= f(t)\, e^{-st} \Big|_{t=0}^{t=h} - \int_0^h f(t)\left(-s e^{-st}\right) dt$$

$$= f(t)\, e^{-sh} - f(0) + s \int_0^h f(t) e^{-st}\, dt$$

By the previous two propositions,

$$f(t) e^{-sh} \to 0 \quad \text{as} \quad h \to \infty \quad \text{for } s > a$$

and $\int_0^h f(t) e^{-st}\, dt$ converges as $h \to \infty$ for $s > a$

Taking the limit as $h \to \infty$ of both sides of the previous expression,

$$\int_0^\infty f'(t) e^{-st}\, dt = s \int_0^\infty f(t) e^{-st}\, dt - f(0),$$

or

$$\mathcal{L}[f'](s) = s\, \mathcal{L}[f](s) - f(0) \quad \text{for all } s > a.$$

Corollary. Suppose that $f, \frac{df}{dt}, \dots, \frac{d^{(r-1)}f}{dt^{(r-1)}}$ are of exponential orders $a_0, a_1, \dots, a_{r-1}$, respectively. Let $a = \max(a_0, \dots, a_r)$. Then

$\mathcal{L}\left[\frac{d^r f}{dt^r}\right]$ converges for $s > a$ and

$$\mathcal{L}\left[\frac{d^r f}{dt^r}\right](s) = s^r \mathcal{L}[f](s) - s^{r-1} f(0) - s^{r-2} \frac{df}{dt}(0) - \dots$$
$$\dots - \frac{d^{r-1} f}{dt^{r-1}}(0)$$

Proof: Proceed by induction on r.

The case $r=1$ is the previous theorem.

Now, suppose the statement is true for $(r-1)$.

Thus,

$\mathcal{L}[f^{(r-1)}]$ converges for $s > \max(a_0, \dots, a_{r-1})$ and

$$\mathcal{L}[f^{(r-1)}](s) = s^{r-1} \mathcal{L}[f](s) - s^{r-2} f(0) - \dots - \frac{d^{r-2} f}{dt^{r-2}}(0)$$

for $s > \max(a_0, \dots, a_{r-1})$.

Then, by the Theorem,

$\mathcal{L}[f^{(r)}] = \mathcal{L}\left[\frac{d}{dt}(f^{(r-1)})\right]$ converges for all

$$s > \max\left(\max(a_0, \dots, a_{r-1}), a_r\right)$$
$$= \max(a_0, \dots, a_r) = a$$

and

$$\mathcal{L}[f^{(r)}](s) = s\mathcal{L}[f^{(r-1)}](s) - \frac{d^{r-1}f}{dt^{r-1}}(0)$$

$$= s\left(s^{r-1}\mathcal{L}[f](s) - s^{r-2}f(0) - \cdots - \frac{d^{r-2}f}{dt^{r-2}}(0)\right) - \frac{d^{r-1}f}{dt^{r-1}}(0)$$

$$= s^r\mathcal{L}[f](s) - s^{r-1}f(0) - \cdots - s\frac{d^{r-2}f}{dt^{r-2}}(0) - \frac{d^{r-1}f}{dt^{r-1}}(0).$$

for all $s > a$. ∎

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We now show how the previous two results, together with the facts that

$$\mathcal{L}[e^{at}](s) = \frac{1}{s-a}, \qquad s > a$$

$$\mathcal{L}[te^{at}](s) = \frac{1}{(s-a)^2}, \qquad s > a$$

$$\mathcal{L}[t^2e^{at}](s) = \frac{2}{(s-a)^3}, \qquad s > a,$$

allow us to find the solutions to the two differential equations from the lecture on the method of undetermined coefficients.

<u>Example</u> $\frac{dy}{dt} + y = te^t$, $y(0) = -\frac{1}{4}$

Solution: Apply $\mathcal{L}$ to both sides

$$\mathcal{L}[y' + y](s) = \mathcal{L}[te^t](s)$$

By linearity of $\mathcal{L}$,

$$\mathcal{L}[y'](s) + \mathcal{L}[y](s) = \frac{1}{(s-1)^2}$$

Now, supposing that the solution y is of exponential order for some a (to be justified on Friday), apply the previous theorem

$$\left(s\mathcal{L}[y](s) - y(0)\right) + \mathcal{L}[y](s) = \frac{1}{(s-1)^2}$$

$$\mathcal{L}[y](s)\left(s+1\right) = y(0) + \frac{1}{(s-1)^2}$$

$$= -\frac{1}{4} + \frac{1}{(s-1)^2}.$$

So,

$$\mathcal{L}[y](s) = -\frac{1}{4}\frac{1}{(s+1)} + \frac{1}{(s+1)(s-1)^2}.$$

Partial fractions:

$$\frac{1}{(s+1)(s-1)^2} = \frac{A}{(s+1)} + \frac{B}{(s-1)} + \frac{C}{(s-1)^2}$$

$$= \frac{A(s-1)^2 + B(s+1)(s-1) + C(s+1)}{(s+1)(s-1)^2}$$

$$= \frac{A(s^2-2s+1) + B(s^2-1) + C(s+1)}{(s+1)(s-1)^2}$$

Matching coefficients of $1, s, s^2$ in numerators of left and right sides,

$$\left.\begin{aligned} A + B &= 0 \\ -2A + C &= 0 \\ A - B + C &= 1 \end{aligned}\right\} \qquad \begin{aligned} A &= -B \\ C &= 2A = -2B \end{aligned}$$

$$1 = A - B + C$$
$$= (-B) - B - 2B = -4B$$

$$B = -\frac{1}{4}, \qquad A = \frac{1}{4}, \qquad C = \frac{1}{2}$$

$$\frac{1}{(s+1)(s-1)^2} = \frac{1}{4}\frac{1}{s+1} - \frac{1}{4}\frac{1}{s-1} + \frac{1}{2}\frac{1}{(s-1)^2}$$

(continued from previous page)

$$\mathcal{L}[y](s) = -\frac{1}{4}\frac{1}{s+1} + \left(\frac{1}{4}\frac{1}{s+1} - \frac{1}{4}\frac{1}{s-1} + \frac{1}{2}\frac{1}{(s-1)^2}\right)$$

$$= -\frac{1}{4}\frac{1}{s-1} + \frac{1}{2}\frac{1}{(s-1)^2}.$$

Therefore,
$$y = \mathcal{L}^{-1}\left[-\frac{1}{4}\frac{1}{s-1} + \frac{1}{2}\frac{1}{(s-1)^2}\right]$$

$$= -\frac{1}{4}\mathcal{L}^{-1}\left[\frac{1}{s-1}\right] + \frac{1}{2}\mathcal{L}^{-1}\left[\frac{1}{(s-1)^2}\right]$$

$$= -\frac{1}{4}e^t + \frac{1}{2}te^t.$$

You can look back through the notes on Undetermined Coefficients to check that we indeed obtained the same solution.

Example. $\frac{dy}{dt} - y = te^t, \qquad y(0) = 0.$

Solution: Apply $\mathcal{L}$ to both sides

$$\mathcal{L}[y'-y](s) = \frac{1}{(s-1)^2}$$

$$\mathcal{L}[y'](s) - \mathcal{L}[y](s) = \frac{1}{(s-1)^2}$$

$$\left(s\mathcal{L}[y](s) - y(0)\right) - \mathcal{L}[y](s) = \frac{1}{(s-1)^2}$$

$$(s-1)\mathcal{L}[y](s) = \frac{1}{(s-1)^2}$$

$$\mathcal{L}[y](s) = \frac{1}{(s-1)^3}$$

Therefore, $y = \mathcal{L}^{-1}\left[\frac{1}{(s-1)^3}\right]$

$$= \mathcal{L}^{-1}\left[\frac{1}{2}\left(\frac{2}{(s-1)^3}\right)\right]$$

$$= \frac{1}{2}\mathcal{L}^{-1}\left[\frac{2}{(s-1)^3}\right]$$

$$= \frac{t^2 e^t}{2}.$$

Again, you can check that this is the same solution as found by undetermined coefficients.

Laplace transform method for solving linear equations with constant coefficients:

$$(L) \quad \frac{d^r y}{dt^r} + a_{r-1}\frac{d^{r-1}y}{dt^{r-1}} + \cdots + a_1\frac{dy}{dt} + a_0 y = F.$$

$$y(0) = \varphi_0^{(0)}, \ldots, \frac{d^{r-1}y}{dt^{r-1}}(0) = \varphi_0^{(r-1)}.$$

1. Take $\mathcal{L}$ of both sides of (L).

2. Solve resulting algebraic equation for $\mathcal{L}[y]$.

3. Take inverse transform to find the solution y