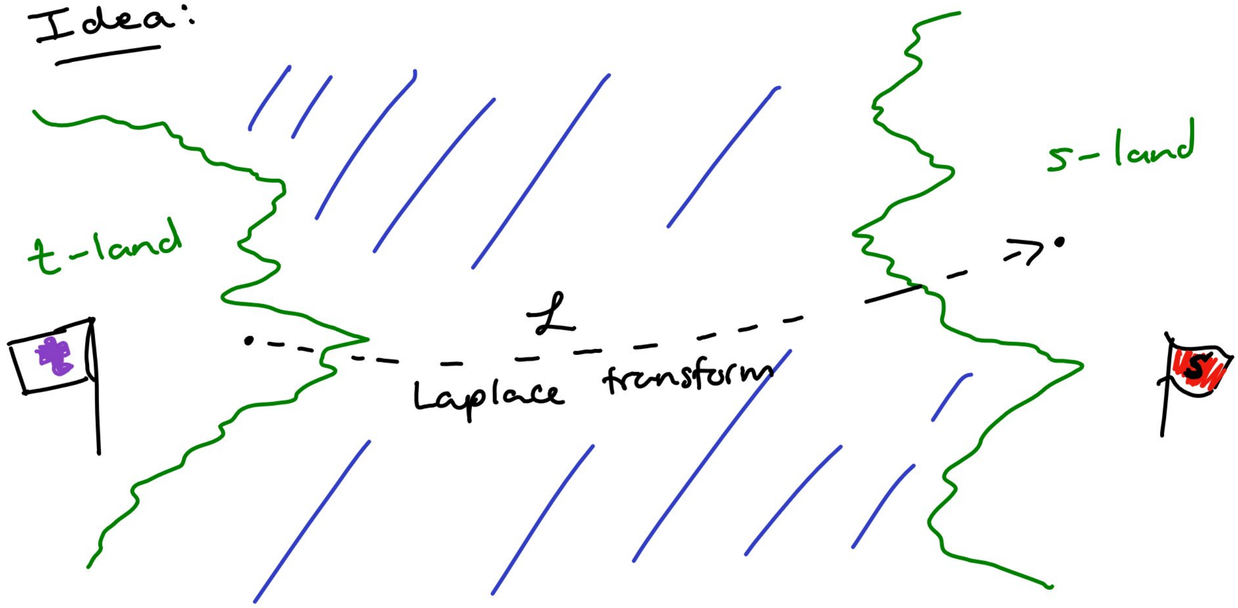
Mthe 237
Lecture 22
Oct. 31, 2017

Topic: Basic Properties of the Laplace Transform

Idea:



The Laplace transform $\mathcal{L}$ (to be defined shortly) is an operator that sends functions to other functions.

$$\begin{pmatrix} \varphi : [0,\infty) \longrightarrow \mathbb{R} \\ t \longmapsto \varphi(t) \end{pmatrix} \xmapsto{\ \mathcal{L}\ } \begin{pmatrix} \mathcal{L}[\varphi] : (\text{Domain}) \longrightarrow \mathbb{R} \\ s \longmapsto \mathcal{L}[\varphi](s) \end{pmatrix}$$

For us, the most important property of the Laplace transform is:

$$\begin{pmatrix} \text{(Linear)} \\ \text{Differential Equations} \\ \text{in } y(t) \end{pmatrix} \xmapsto{\ \mathcal{L}\ } \begin{pmatrix} \underline{\text{Algebraic}} \text{ Equations} \\ \text{in } \mathcal{L}[y](s) \end{pmatrix}$$

Typically, algebraic equations are simpler to solve. This gives rise to another technique for solving linear differential equations.

As an example, the differential equation

$$\frac{d^2 y}{dt^2} + y = 0 \ , \quad \begin{array}{l} y(0) = 1 \\ y'(0) = 0 \end{array}$$

gets transformed to the algebraic equation

$$\left(s^2 \mathcal{L}[y](s) - s\right) + \mathcal{L}[y] = 0,$$

which we can solve to get

$$\mathcal{L}[y](s) = \frac{s}{s^2+1} .$$

Moreover, when restricted to "good" families of functions (and we shall see an example of one such good family), $\mathcal{L}$ is an <u>injective</u> operation.

This means that, once we compute that

$$\mathcal{L}[\cos(\omega t)](s) = \frac{s}{s^2+\omega^2},$$

we can read off that the solution to the differential equation we started with is

$$y(t) = \cos(t).$$

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§ Improper Integrals

Suppose that $f: [c, \infty) \to \mathbb{R}$ is an integrable (for example, continuous) function.

If the limit $\lim_{h \to \infty} \int_c^h f(t)\, dt$ exists, we say that the improper integral of f converges, and denote the limit by $\int_c^\infty f(t)\, dt$.

Otherwise, we say that the improper integral of $f(t)$ diverges.

Analogy: Finite Sums $\sum_{k=a}^{n} a_k$ $\updownarrow$ Integrals $\int_a^b f(t)\,dt$ — Infinite Series $\sum_{k=a}^{\infty} a_k$ $\updownarrow$ Improper Integrals $\int_a^\infty f(t)\,dt$

Examples:

- $$\int_1^h \frac{dt}{t} = \ln(t)\Big|_1^h = \ln(h) - \ln(1) = \ln(h)$$

  As $h \to \infty$, $\ln(h) \to \infty$. So $\int_1^\infty \frac{dt}{t}$ diverges.

- $$\int_0^h \frac{dt}{t^2+1} = \arctan(t)\Big|_0^h = \arctan(h) - \arctan(0) = \arctan(h)$$

  As $h \to \infty$, $\arctan(h) \to \frac{\pi}{2}$.

  So $$\int_0^\infty \frac{dt}{t^2+1} = \frac{\pi}{2}.$$

- $\int_0^h \cos(t)\,dt = \sin(t)\Big|_0^h = \sin(h) - \sin(0)$
$= \sin(h).$

As $h \to \infty$, $\sin(h)$ oscillates between -1 and 1.

The limit does not exist and $\int_0^\infty \cos(t)\,dt$ diverges.

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§ The Laplace Transform

Def. Let $f : [0,\infty) \to \mathbb{R}$ be integrable. The Laplace transform $\mathcal{L}[f]$ of f is the function

$$\mathcal{L}[f] : s \longmapsto \int_0^\infty f(t)\,e^{-st}\,dt$$

with domain

$$\left\{ s \in \mathbb{R} : \int_0^\infty f(t)\,e^{-st}\,dt \ \text{ converges} \right\}$$

(Note the domain could be the empty set!)

Remark. It is possible to define a more general version of the Laplace transform that takes $f : [0,\infty) \to \mathbb{C}$ to

$$z \longmapsto \int_0^\infty f(t)\,e^{-zt}\,dt, \quad z \in \mathbb{C}.$$

However, the real version will be sufficient for the purposes of this course.

<u>Examples:</u>

- $f(t) = 1$ for all $t \in [0, \infty)$.

$$\int_0^h f(t)\, e^{-st}\, dt = \int_0^h e^{-st}\, dt$$

$$\overset{s \neq 0}{=} -\frac{1}{s} e^{-st} \Big|_{t=0}^{t=h}$$

$$= \left(-\frac{e^{-sh}}{s} + \frac{1}{s} \right)$$

$$\xrightarrow[h \to \infty]{} \begin{cases} 0 + \frac{1}{s}, & s > 0 \\ \text{Diverges to } \infty, & s < 0 \end{cases}$$

(Reminder: graph of e^{ax} looks like

$a > 0$ $a < 0$)

Moreover, when $s = 0$, $\int_0^h e^{-st}\, dt = \int_0^h dt = h \to \infty$ as $h \to \infty$

So $\mathcal{L}[1](s) = \frac{1}{s}$, $s > 0$

- $f(t) = t$ for all $t \in [0, \infty)$.

$$\int_0^h \underbrace{t}_{u}\,\underbrace{e^{-st}}_{v'}\,dt \underset{s \neq 0}{=} -\frac{1}{s} t e^{-st} \Big|_{t=0}^{t=h} - \int_0^h \left(-\frac{1}{s} e^{-st}\right) dt$$

$$= -\frac{1}{s} t e^{-st} \Big|_{t=0}^{t=h} - \frac{1}{s^2} e^{-st} \Big|_{t=0}^{t=h}$$

$$= \left(-\frac{1}{s} h e^{-sh} + 0\right) - \left(\frac{1}{s^2} e^{-sh} - \frac{1}{s^2}\right)$$

$$\xrightarrow[h \to \infty]{} \begin{cases} 0 + 0 + \frac{1}{s^2}, & s > 0 \\ \text{Diverges to } \infty, & s < 0 \end{cases}$$

When $s = 0$, $\displaystyle\int_0^h t\,dt = \frac{t^2}{2}\Big|_0^h \longrightarrow \infty$ as $h \to \infty$.

$$\mathcal{L}[t](s) = \frac{1}{s^2}, \qquad s > 0.$$

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§ Domain

Q: What does the domain of $\mathcal{L}[f]$ look like in general?

A: The answer follows from the following theorem, which we simply state without proof:

Theorem If $\int_0^\infty f(t)\, e^{-st}\, dt$ converges for $s = s_0$, then it converges for every $s > s_0$.

Reference: Doetsch, Introduction to the Theory and Application of the Laplace Transformation, Th. 3.4, pp. 16–17.

Corollary The domain of $\mathcal{L}[f]$ is equal to one of the following:

- $\emptyset$ (Empty set)
- $s > a$
- $s \geq a$

$\Big\}$ For some real a

- $\mathbb{R}$

(The complex case is similar, there the possibilities are
$\emptyset$, $Re(z) > a$, $Re(z) \geq a$ and $\mathbb{C}$.)

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§ Linearity

Prop. If $\mathcal{L}[f_1]$ converges for $s > s_1$ and $\mathcal{L}[f_2]$ converges for $s > s_2$, then

for any $c_1, c_2 \in \mathbb{R}$,

$\mathcal{L}[c_1 f_1 + c_2 f_2]$ converges for $s > \max(s_1, s_2)$

and $\mathcal{L}[c_1 f_1 + c_2 f_2](s) = c_1 \mathcal{L}[f_1](s) + c_2 \mathcal{L}[f_2](s)$

Proof: This follows from linearity of lim and $\int$.

$$\int_0^h (c_1 f_1 + c_2 f_2)(t) e^{-st}\, dt$$

$$= c_1 \int_0^h f_1(t) e^{-st}\, dt + c_2 \int_0^h f_2(t) e^{-st}\, dt$$

By hypothesis, $\lim_{h \to \infty} \int_0^h f_1(t) e^{-st}\, dt$ and

$\lim_{h \to \infty} \int_0^h f_2(t) e^{-st}\, dt$

exist for $s > \max(s_1, s_2)$.

Therefore, so does $\lim_{h \to \infty} \left(c_1 \int_0^h f_1 e^{-st}\, dt + c_2 \int_0^h f_2 e^{-st}\, dt \right)$

$$= \int_0^\infty (c_1 f_1 + c_2 f_2)(t) e^{-st}\, dt$$

and

$$\lim_{h\to\infty}\left(c_1\int_0^h f_1 e^{-st}\,dt + c_2\int_0^h f_2 e^{-st}\,dt \right)$$

$$= c_1 \lim_{h\to\infty}\int_0^h f_1 e^{-st}\,dt + c_2 \lim_{h\to\infty}\int_0^h f_2 e^{-st}\,dt.$$

$$= c_1\,\mathcal{L}[f_1](s) + c_2\,\mathcal{L}[f_2](s), \qquad s > \max(s_1, s_2).$$

$\square$

Example: $$\mathcal{L}\left[\pi + et\right](s) = \pi\,\mathcal{L}[1](s) + e\,\mathcal{L}[t](s)$$

$$= \frac{\pi}{s} + \frac{e}{s^2} = \frac{\pi s + e}{s^2}, \qquad s > 0.$$

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Injectivity

Another important theorem that we state without proof is

> **Theorem.** If there exists $a \in \mathbb{R}$ so that
> $$\mathcal{L}[f_1](s) = \mathcal{L}[f_2](s) \quad \text{for all } s > a,$$
> and the functions f_1, f_2 are continuous,
> then $f_1 = f_2$.

Reference: ibid (Doetsch) § 5, pp. 20–24

In other words, $\mathcal{L}$ is an <u>injective</u> operation when restricted to continuous functions with nonempty Laplace transforms!

This lets us unambiguously define:

<u>Def.</u> If $F = \mathcal{L}[f]$, where f is continuous, we say that f is <u>the</u> inverse Laplace transform of F:

$$f = \mathcal{L}^{-1}[F].$$

<u>Examples</u>: $\mathcal{L}^{-1}\left[\frac{1}{s}\right] = 1$, $\mathcal{L}^{-1}\left[\frac{1}{s^2}\right] = t$.

<u>Prop.</u> For any $c_1, c_2 \in \mathbb{R}$, and $F_1 = \mathcal{L}[f_1]$, $F_2 = \mathcal{L}[f_2]$ with f_1, f_2 continuous,

$$\mathcal{L}^{-1}[c_1 F_1 + c_2 F_2] = c_1 \mathcal{L}^{-1}[F_1] + c_2 \mathcal{L}^{-1}[F_2].$$

<u>Proof</u>: $c_1 f_1 + c_2 f_2$ is continuous, and by linearity of $\mathcal{L}$, we have

$$\begin{aligned}\mathcal{L}[c_1 f_1 + c_2 f_2] &= c_1 \mathcal{L}[f_1] + c_2 \mathcal{L}[f_2] \\ &= c_1 F_1 + c_2 F_2.\end{aligned}$$

Thus, $\mathcal{L}^{-1}[c_1F_1 + c_2F_2] = c_1f_1 + c_2f_2$

$$= c_1\mathcal{L}^{-1}[F_1] + c_2\mathcal{L}^{-1}[F_2].$$

Unimportant
Remark: For the complex version of the Laplace transform, there is an expression for the inverse transform

$$\mathcal{L}^{-1}[F](t) = \frac{e^{\sigma t}}{2\pi}\int_{-\infty}^{\infty} F(\sigma + i\omega)e^{i\omega t}\,dt.$$

However, this may not converge, and we will not develop this expression or use it further.