Mthe 237
Lecture 21
Oct. 27, 2017

Topic: Variation of Parameters (Arbitrary Order)

Today we see how the method of variation of parameters works for linear equations of any order. An interesting special case is first order equations.

The equation we are to solve is

$$(L) \qquad \frac{d^r y}{dt^r} + a_{r-1}(t)\frac{d^{r-1}y}{dt^{r-1}} + \cdots + a_1(t)\frac{dy}{dt} + a_0(t)y = F(t)$$

Its associated homogeneous equation is

$$(H_L) \qquad \frac{d^r y}{dt^r} + a_{r-1}(t)\frac{d^{r-1}y}{dt^{r-1}} + \cdots + a_1(t)\frac{dy}{dt} + a_0(t)y = 0.$$

Suppose that $\varphi_1, \dots, \varphi_r$ are linearly independent solutions of (H_L) over $I \subset \mathbb{R}$.

Theorem. Suppose that $u_1, \dots, u_r$ are differentiable functions over I. If the derivatives $u_1', \dots, u_r'$ satisfy the system of linear equations

$$\begin{pmatrix} \varphi_1 & \cdots & \varphi_r \\ \varphi_1' & \cdots & \varphi_r' \\ \vdots & \ddots & \vdots \\ \varphi_1^{(r-1)} & \cdots & \varphi_r^{(r-1)} \end{pmatrix} \begin{pmatrix} u_1' \\ u_2' \\ \vdots \\ u_r' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ F \end{pmatrix} \qquad (*)$$

(Here the notation means $y^{(j)} = \frac{d^j y}{dt^j}$.)

Then the function

$$\varphi_p = u_1 \varphi_1 + \cdots + u_r \varphi_r$$

is a particular solution over I.

Proof: As for 2nd order, this comes down to a computation.

$$\varphi_p = \sum_j u_j \varphi_j$$

$$\varphi_p' = \sum_j \left(u_j' \varphi_j + u_j \varphi_j'\right)$$

$$= \underbrace{\left(\sum_j u_j' \varphi_j\right)}_{=0 \text{ by first row of } (*)} + \left(\sum_j u_j \varphi_j'\right)$$

$$= \sum_j u_j \varphi_j'$$

$$\varphi_p'' = \sum_j \left(u_j' \varphi_j' + u_j \varphi_j''\right)$$

$$= \underbrace{\left(\sum_j u_j' \varphi_j'\right)}_{=0 \text{ by second row of } (*)} + \left(\sum_j u_j \varphi_j''\right)$$

$$= \sum_j u_j \varphi_j''$$

$$\vdots$$

$$\varphi_p^{(r-1)} = \sum_j u_j \varphi_j^{(r-1)}$$

$$\varphi_p^{(r)} = \underbrace{\left(\sum_j u_j' \varphi_j^{(r-1)}\right)}_{= F} + \left(\sum_j u_j \varphi_j^{(r)}\right)$$

So

$$\varphi_p^{(r)} + a_{r-1}\varphi_p^{(r-1)} + \cdots + a_1 \varphi_p' + a_0 \varphi_p$$

$$= F + \left(\sum_j u_j \varphi_j^{(r)}\right) + a_{r-1}\left(\sum_j u_j \varphi_j^{(r-1)}\right) + \cdots + a_1\left(\sum_j u_j \varphi_j'\right) + a_0\left(\sum_j u_j \varphi_j\right)$$

$$= F + \sum_j u_j \Big(\underbrace{\varphi_j^{(r)} + a_{r-1}\varphi_j^{(r-1)} + \cdots + a_1 \varphi_j' + a_0 \varphi_j}_{=0 \text{ because } \varphi_j \text{ solves } (H_L)} \Big)$$

$$= F$$

$\blacksquare$

The system of equations $(*)$ can always be solved for $u_1', \ldots, u_r'$, because $\varphi_1, \ldots, \varphi_r$ are linearly independent and therefore have a nonvanishing Wronskian.

In practice, one of the more efficient ways of solving $(*)$ is Gaussian elimination. However, here we would like to present another interesting way of solving $(*)$.

Cramer's Rule:

Let A be a $r \times r$ matrix with $\det A \neq 0$, and suppose we are trying to solve the system of linear equations

$$Ax = y \qquad (x, y \in \mathbb{R}^r)$$

Let A_j be the matrix obtained by replacing the j^{th} column of A by y.

Theorem (Cramer's Rule)

$$x_j = \frac{\det A_j}{\det A}, \quad \text{for each } j = 1, \ldots, r.$$

Unwinding Cramer's Rule for $r=1$ and 2:

$\underline{r=1}$: The matrix equation

$$(a)(x_1) = (y_1)$$

is equivalent to the algebraic equation

(by def. of matrix multiplication)

$$ax_1 = y_1$$

$\det(a) \neq 0 \Leftrightarrow a \neq 0$. So we can divide both sides by a.

$$x_1 = \frac{y_1}{a} = \frac{\det(y_1)}{\det(a)} = \frac{\det A_1}{\det A}.$$

$\underline{r=2}$: $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$.

Cramer's rule says:

$$x_1 = \frac{\det\begin{pmatrix} y_1 & b \\ y_2 & d \end{pmatrix}}{\det\begin{pmatrix} a & b \\ c & d \end{pmatrix}} = \frac{y_1 d - y_2 b}{ad-bc},$$

$$x_2 = \frac{\det\begin{pmatrix} a & y_1 \\ c & y_2 \end{pmatrix}}{\det\begin{pmatrix} a & b \\ c & d \end{pmatrix}} = \frac{y_2 a - y_1 c}{ad-bc}.$$

Comparing this with what we know about

$$A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix},$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$$= \frac{1}{ad-bc}\begin{pmatrix} dy_1 - by_2 \\ -cy_1 + ay_2 \end{pmatrix}$$

so that

$$x_1 = \frac{dy_1 - by_2}{ad-bc},$$

$$x_2 = \frac{ay_2 - cy_1}{ad-bc},$$ in agreement with Cramer's rule.

<u>Example</u>

$$\left(\frac{d}{dt}-1\right)\left(\frac{d}{dt}-2\right)\left(\frac{d}{dt}-3\right)y = \sin(e^{-t})$$

The characteristic polynomial of the associated homogeneous equation is

$$\chi(z) = (z-1)(z-2)(z-3).$$

Basis of solutions of associated homogeneous is $\{e^t, e^{2t}, e^{3t}\}$.

Take $\varphi_1 = e^t$, $\varphi_2 = e^{2t}$, $\varphi_3 = e^{3t}$.

$$W(\varphi_1, \varphi_2, \varphi_3)(t) = \det \begin{pmatrix} e^t & e^{2t} & e^{3t} \\ e^t & 2e^{2t} & 3e^{3t} \\ e^t & 4e^{2t} & 9e^{3t} \end{pmatrix}$$

$$= e^t e^{2t} e^{3t} \det \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{pmatrix}$$

$$= e^{6t} \det \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 3 & 8 \end{pmatrix}$$

$$= e^{6t} \cdot 1 \cdot \det \begin{pmatrix} 1 & 2 \\ 3 & 8 \end{pmatrix}$$

$$= 2e^{6t}.$$

The derivatives u_1', u_2', u_3' satisfy

$$\begin{pmatrix} e^t & e^{2t} & e^{3t} \\ e^t & 2e^{2t} & 3e^{3t} \\ e^t & 4e^{2t} & 9e^{3t} \end{pmatrix} \begin{pmatrix} u_1' \\ u_2' \\ u_3' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \sin(e^{-t}) \end{pmatrix}.$$

We solve this using Cramer's rule:

$$u_1' = \frac{\det\begin{pmatrix} 0 & e^{2t} & e^{3t} \\ 0 & 2e^{2t} & 3e^{3t} \\ \sin(e^{-t}) & 4e^{2t} & 9e^{3t} \end{pmatrix}}{W}$$

$$= \frac{\sin(e^{-t}) \det\begin{pmatrix} e^{2t} & e^{3t} \\ 2e^{2t} & 3e^{3t} \end{pmatrix}}{2e^{6t}}$$

$$= \frac{\sin(e^{-t})\,(3e^{5t} - 2e^{5t})}{2e^{6t}}$$

$$= \frac{1}{2} e^{-t} \sin(e^{-t}).$$

$$u_1 = -\frac{1}{2}\int -e^{-t}\sin(e^{-t})\,dt$$

$$\left|\begin{array}{l} s = e^{-t} \\ ds = -e^{-t}\,dt \end{array}\right.$$

$$= -\frac{1}{2}\int \sin(s)\,ds$$

$$= \frac{1}{2}\cos(s)$$

So,

$$u_1 = \frac{1}{2}\cos(e^{-t})$$

$$u_2' = \frac{\det\begin{pmatrix} e^{t} & 0 & e^{3t} \\ e^{t} & 0 & 3e^{3t} \\ e^{t} & \sin(e^{-t}) & 9e^{3t} \end{pmatrix}}{W}$$

$$= \frac{-\sin(e^{-t}) \det\begin{pmatrix} e^{t} & e^{3t} \\ e^{t} & 3e^{3t} \end{pmatrix}}{2e^{6t}}$$

$$= \frac{-\sin(e^{-t})\,(3e^{4t} - e^{4t})}{2e^{6t}}$$

$$= -e^{-2t}\sin(e^{-t})$$

So,

$$u_2 = -e^{-t}\cos(e^{-t}) + \sin(e^{-t})$$

$$\int -e^{-t}e^{-t}\sin(e^{-t})\,dt$$

$$= \int s\sin(s)\,ds \quad (s=e^{-t})$$

$$= s(-\cos(s)) - \int(-\cos(s))\,ds$$

$$= -s\cos(s) + \sin(s)$$

$$= -e^{-t}\cos(e^{-t}) + \sin(e^{-t})$$

$$u_3' = \frac{\det\begin{pmatrix} e^t & e^{2t} & 0 \\ e^t & 2e^{2t} & 0 \\ e^t & 4e^{2t} & \sin(e^{-t}) \end{pmatrix}}{W}$$

$$= \frac{\sin(e^{-t})\det\begin{pmatrix} e^t & e^{2t} \\ e^t & 2e^{2t} \end{pmatrix}}{2e^{6t}}$$

$$= \frac{1}{2}e^{-3t}\sin(e^{-t})$$

So,

$$u_3 = \frac{1}{2}e^{-2t}\cos(e^{-t}) - e^{-t}\sin(e^{-t}) - \cos(e^{-t})$$

$$-\frac{1}{2}\int -e^{-2t}e^{-t}\sin(e^{-t})\,dt$$

$$= -\frac{1}{2}\int s^2\sin(s)\,ds$$

Now,

$$\int s^2\sin(s)\,ds$$

$$= s^2(-\cos(s)) - \int 2s(-\cos(s))\,ds$$

$$= -s^2\cos(s) + 2\int s\cos(s)\,ds$$

$$= -s^2\cos(s) + 2\left(s\sin(s) - \int\sin(s)\,ds\right)$$

$$= -s^2\cos(s) + 2s\sin(s) + 2\cos(s)$$

$$\varphi_p = u_1\varphi_1 + u_2\varphi_2 + u_3\varphi_3$$

$$= \left(\tfrac{1}{2}\cos(e^{-t})\right)e^{t} + \left(\sin(e^{-t}) - e^{-t}\cos(e^{-t})\right)e^{2t}$$

$$+ \left(\tfrac{1}{2}e^{-2t}\cos(e^{-t}) - e^{-t}\sin(e^{-t}) - \cos(e^{-t})\right)e^{3t}$$

$$= \frac{e^{t}}{2}\cos(e^{-t}) + e^{2t}\sin(e^{-t}) - e^{t}\cos(e^{-t})$$

$$+ \tfrac{1}{2}e^{t}\cos(e^{-t}) - e^{2t}\sin(e^{-t}) - e^{3t}\cos(e^{-t})$$

$= -e^{3t}\cos(e^{-t})$ is the particular solution.

The affine space of all solutions is

$$-e^{3t}\cos(e^{-t}) + c_1 e^{t} + c_2 e^{2t} + c_3 e^{3t},$$

$$c_1, c_2, c_3 \in \mathbb{R}.$$

FIRST ORDER LINEAR EQUATIONS

These can <u>always</u> be solved (at least, up to computing an integral), even in the time-varying coefficients case.

$$(L) \qquad \frac{dy}{dt} + a(t)\, y = F(t).$$

The associated homogeneous equation is

$$(H_L) \qquad \frac{dy}{dt} + a(t)\, y = 0.$$

Now, (H_L) is separable:

$$\left.\begin{aligned} \frac{1}{y}\frac{dy}{dt} &= -a(t) \\ \ln|y| &= -\int a(t)\,dt \\ y(t) &= C \exp\left(-\int a(t)\,dt\right). \end{aligned}\right\}$$

Interesting on its own

- can always solve $\frac{dy}{dt} + a(t)y = 0.$

For use in variation of parameters, can take $C=1$.

$$\varphi = \exp\left(-\int a(t)\,dt\right).$$

Suppose $u'\varphi = F$ (this is the 1×1 version of the system of equations (*) from earlier)

Then $\varphi_p = u\varphi$ is a particular solution of (L).

$$u' = \frac{F}{\varphi} = \frac{F}{\exp\left(-\int a(t)\,dt\right)} = F \exp\left(\int a(t)\,dt\right)$$

$$u = \int \left[F(t) \exp\left(\int a(t)\, dt\right) \right] dt$$

$$\varphi_p = \exp\left(-\int a(t)\, dt\right) \int \left[F(t) \exp\left(\int a(t)\, dt\right) \right] dt$$

All solutions then have the form

$$\varphi_p + c \exp\left(-\int a(t)\, dt\right), \qquad c \in \mathbb{R}.$$

It may not be worth remembering these fairly complicated explicit formulas – instead, it may be better to work through variation of parameters in each case.

<u>Examples</u>.

- $$\frac{dy}{dt} - 2ty = e^{t^2}.$$

Associated homogeneous:

$$\frac{dy}{dt} - 2ty = 0, \qquad \frac{dy}{dt} = 2ty$$

$$\frac{1}{y} \frac{dy}{dt} = 2t$$

$$\ln|y| = t^2$$

$$y = e^{t^2}. \qquad \text{Take } \varphi = e^{t^2}.$$

$$u' = \frac{F}{\varphi} = \frac{e^{t^2}}{e^{t^2}} = 1 \implies u = t.$$

$$\varphi_p = u\varphi = te^{t^2}$$

All solutions look like $te^{t^2} + ce^{t^2}$, $c \in \mathbb{R}$

$$= (t+c)e^{t^2}$$

- $t^3 \frac{dy}{dt} + 3t^2 y = \sin(t)$, $\quad t > 0$

Convert to standard form:

$$\frac{dy}{dt} + \frac{3}{t} y = \frac{\sin(t)}{t^3}$$

Associated homogeneous equation is

$$\frac{dy}{dt} + \frac{3}{t} y = 0$$

$$\frac{1}{y}\frac{dy}{dt} = -\frac{3}{t}$$

$$\ln|y| = \underset{(+c)}{-3\ln(t)} = \ln(t^{-3})\ (+c)$$

$y = \frac{c}{t^3}$. Take $\varphi = \frac{1}{t^3}$.

$$u' = \frac{F}{\varphi} = \frac{\sin(t)/t^3}{1/t^3} = \sin(t)$$

So, $u = -\cos(t)$

$$\varphi_p = (-\cos(t))\,\frac{1}{t^3}.$$

All solutions look like

$$-\frac{\cos(t)}{t^3} + \frac{c}{t^3}, \qquad c \in \mathbb{R}.$$