MTHE 237
Lecture 20
Oct. 25, 2017

Topic: Variation of Parameters (Second-Order)

Today we will develop another method of solving nonhomogeneous linear differential equations, called <u>variation of parameters</u>.

A major advantage of variation of parameters is that it does not require special assumptions about the forcing term $F(t)$.

(On the other hand, the method of undetermined coefficients only applies when F is a quasi-polynomial.)

Today, we will focus on second-order equations. The method extends to equations of arbitrary order, which we will see in the next lecture.

$$(L) \qquad \frac{d^2y}{dt^2} + a_1(t)\frac{dy}{dt} + a_0(t)y = F(t).$$

The associated homogeneous equation is

$$(L_H) \qquad \frac{d^2y}{dt^2} + a_1(t)\frac{dy}{dt} + a_0(t)y = 0.$$

Suppose that φ_1 and φ_2 are linearly independent solutions of (L_H) [when (L) has constant coefficients, we have a general method for finding φ_1, φ_2] over an interval I.

By linearity, $c_1\varphi_1 + c_2\varphi_2$, $c_1, c_2 \in \mathbb{R}$ is again a solution of (L_H) over I.

Idea of method (due to Lagrange):

Look for a particular solution of (L) of the form

$$u_1(t)\varphi_1(t) + u_2(t)\varphi_2(t).$$

u_1 and u_2 are called varying parameters.

Theorem. Suppose that u_1, u_2 are differentiable and their derivatives u_1', u_2' satisfy the system of equations

$$\left.\begin{aligned} (1)\quad & u_1'\varphi_1 + u_2'\varphi_2 = 0 \\ (2)\quad & u_1'\varphi_1' + u_2'\varphi_2' = F \end{aligned}\right\}$$

Then the function

$$\varphi_p = u_1\varphi_1 + u_2\varphi_2$$

is a solution of (L).

Proof: This is a computation (and the conditions (1) and (2) were likely discovered by doing such a computation first).

$$\varphi_p = u_1\varphi_1 + u_2\varphi_2$$

$$\begin{aligned}
\varphi_p' &= (u_1'\varphi_1 + u_1\varphi_1') + (u_2'\varphi_2 + u_2\varphi_2') \\
&= (\underbrace{u_1'\varphi_1 + u_2'\varphi_2}_{=0 \text{ by } (1)}) + (u_1\varphi_1' + u_2\varphi_2') \\
&= u_1\varphi_1' + u_2\varphi_2'
\end{aligned}$$

$$\varphi_p'' = (\underbrace{u_1'\varphi_1' + u_2'\varphi_2'}_{=F \text{ by } (2)}) + (u_1\varphi_1'' + u_2\varphi_2'')$$

Then,

$$\begin{aligned}
&\varphi_p'' + a_1\varphi_p' + a_0\varphi_p \\
&= F + (u_1\varphi_1'' + u_2\varphi_2'') + a_1(u_1\varphi_1' + u_2\varphi_2') \\
&\qquad + a_0(u_1\varphi_1 + u_2\varphi_2) \\
&= F + u_1(\underbrace{\varphi_1'' + a_1\varphi_1' + a_0\varphi_1}_{=0}) + u_2(\underbrace{\varphi_2'' + a_1\varphi_2' + a_0\varphi_2}_{=0}) \\
&= F.
\end{aligned}$$

because φ_1, φ_2 are solutions of (H_L). $\square$

The system of equations for u_1', u_2' may be written in matrix form as:

$$\underbrace{\begin{pmatrix} \varphi_1 & \varphi_2 \\ \varphi_1' & \varphi_2' \end{pmatrix}}\begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 0 \\ F \end{pmatrix}$$

The determinant of this matrix is the Wronskian of φ_1 and φ_2. Because φ_1 and φ_2 are linearly independent, $W(\varphi_1, \varphi_2)(t) \neq 0$ for all $t \in I$. So the matrix is invertible.

Reminder: Inverse of a 2×2 matrix is

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

So we have

$$\begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \frac{1}{W(\varphi_1, \varphi_2)(t)}\begin{pmatrix} \varphi_2' & -\varphi_2 \\ -\varphi_1' & \varphi_1 \end{pmatrix}\begin{pmatrix} 0 \\ F \end{pmatrix}$$

$$u_1' = -\frac{\varphi_2 F}{W} \implies u_1 = \int -\frac{\varphi_2 F}{W}\, dt$$

$$u_2' = \frac{\varphi_1 F}{W} \implies u_2 = \int \frac{\varphi_1 F}{W}\, dt$$

So we have found the following expression for a particular solution:

$$\varphi_p = \varphi_1 \int -\frac{\varphi_2 F}{W}\, dt + \varphi_2 \int \frac{\varphi_1 F}{W}\, dt.$$

The affine space of all solutions of (L) is then

$$\varphi_1 \int -\frac{\varphi_2 F}{W}\, dt + \varphi_2 \int \frac{\varphi_1 F}{W}\, dt + c_1\varphi_1 + c_2\varphi_2,$$

$$c_1, c_2 \in \mathbb{R}.$$

EXAMPLES

- $\dfrac{d^2y}{dt^2} - 3\dfrac{dy}{dt} + 2y = \sin(e^{-t}).$

  Associated homogeneous equation:

  $$\frac{d^2y}{dt^2} - 3\frac{dy}{dt} + 2y = 0$$

  $$\chi(z) = z^2 - 3z + 2$$
  $$= (z-1)(z-2)$$

  Basis of solutions: $\{e^t, e^{2t}\}$

  Take $\varphi_1(t) = e^t$, $\varphi_2(t) = e^{2t}$.

$$W(\varphi_1, \varphi_2)(t) = \det \begin{pmatrix} e^t & e^{2t} \\ e^t & 2e^{2t} \end{pmatrix}$$

$$= 2e^{3t} - e^{3t} = e^{3t}.$$

$$u_1' = -\frac{\varphi_2 F}{W} = -\frac{e^{2t} \sin(e^{-t})}{e^{3t}} = -e^{-t} \sin(e^{-t}).$$

$$u_1 = \int -e^{-t} \sin(e^{-t})\, dt$$

let $s = e^{-t}$
$ds = -e^{-t} dt$

$$\int \sin(s)\, ds = -\cos(s) = -\cos(e^{-t})$$

$$u_2' = \frac{\varphi_1 F}{W} = \frac{e^t \sin(e^{-t})}{e^{3t}} = e^{-2t} \sin(e^{-t})$$

$$= e^{-t} \left(e^{-t} \sin(e^{-t})\right)$$

$$u_2 = \int e^{-t} \left(e^{-t} \sin(e^{-t})\right) dt$$

let $s = e^{-t}$
$ds = -e^{-t} dt$

$$= -\int s \cdot \sin(s)\, ds$$

$u = s$	$du = ds$
$dv = \sin(s)\, ds$	$v = -\cos(s)$

$$= -\left[-s\cos(s) - \int (-\cos(s))\, ds \right]$$

$$= -\left[-s\cos(s) + \sin(s) \right] = s\cos(s) - \sin(s)$$

$$= e^{-t}\cos(e^{-t}) - \sin(e^{-t})$$

$$\varphi_p = u_1\varphi_1 + u_2\varphi_2$$

$$= (-\cos(e^{-t}))e^{t} + \left(e^{-t}\cos(e^{-t}) - \sin(e^{-t})\right)e^{2t}$$

$$= -\sin(e^{-t})e^{2t}$$

Affine space of all solutions is

$$-\sin(e^{-t})e^{2t} + c_1 e^{t} + c_2 e^{2t}, \qquad c_1, c_2 \in \mathbb{R}.$$

Example of variation of parameters applied to differential equation with time-varying coefficients:

$$\frac{d^2y}{dt^2} + \frac{1}{t}\frac{dy}{dt} - \frac{1}{t^2}y = \frac{1}{t}, \quad t > 0.$$

Claim: $\varphi_1 = t$ and $\varphi_2 = \frac{1}{t}$ are solutions of the associated homogeneous equation

$$\frac{d^2y}{dt^2} + \frac{1}{t}\frac{dy}{dt} - \frac{1}{t^2}y = 0.$$

Check: $\varphi_1' = 1, \qquad \varphi_1'' = 0.$

$$0 + \frac{1}{t}\cdot 1 - \frac{1}{t^2}\cdot t = 0.$$

$$\varphi_2' = -\frac{1}{t^2}, \qquad \varphi_2'' = \frac{2}{t^3}$$

$$\frac{2}{t^3} + \frac{1}{t}\left(-\frac{1}{t^2}\right) - \frac{1}{t^2}\left(\frac{1}{t}\right) = 0.$$

$$W(\varphi_1,\varphi_2)(t) = \det\begin{pmatrix} t & 1/t \\ 1 & -1/t^2 \end{pmatrix}$$

$$= -1/t - 1/t = -2/t$$

This is not zero for all $t>0$, which verifies that φ_1 and φ_2 are linearly independent.

$$u_1' = -\frac{\varphi_2 F}{W} = \frac{-(1/t)(1/t)}{(-2/t)} = \frac{1}{2t}.$$

$$u_1 = \int \frac{dt}{2t} = \frac{1}{2}\int\frac{dt}{t} = \frac{1}{2}\ln(t)$$

$$u_2' = \frac{\varphi_1 F}{W} = \frac{t(1/t)}{(-2/t)} = -\frac{t}{2}$$

$$u_1 = \int -\frac{t}{2}\,dt = -\frac{t^2}{4}.$$

$$\varphi_p = u_1\varphi_1 + u_2\varphi_2$$

$$= \frac{t}{2}\ln(t) - \frac{t}{4}$$

All solutions: $\frac{t}{2}\ln(t) - \frac{t}{4} + c_1 t + \frac{c_2}{t}$, $c_1, c_2 \in \mathbb{R}$.

More examples not seen in lecture:

- $$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 4y = 3te^{-2t}.$$

Associated homogeneous equation:

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 4y = 0.$$

$$\chi(z) = z^2 + 4z + 4 = (z+2)^2$$

Basis : $\{e^{-2t}, te^{-2t}\}$

Take $\varphi_1 = e^{-2t}$, $\varphi_2 = te^{-2t}$.

$$W(\varphi_1, \varphi_2)(t) = \det\begin{pmatrix} e^{-2t} & te^{-2t} \\ -2e^{-2t} & (1-2t)e^{-2t} \end{pmatrix}$$

$$= (1-2t)e^{-4t} + 2te^{-4t}$$

$$= e^{-4t}$$

$$u_1' = \frac{-\varphi_2 F}{W} = \frac{-te^{-2t}\left(3te^{-2t}\right)}{e^{-4t}} = -3t^2$$

$$u_1(t) = \int -3t^2\,dt = -t^3$$

$$u_2' = \frac{\varphi_1 F}{W} = \frac{e^{-2t}\left(3te^{-2t}\right)}{e^{-4t}} = 3t$$

$$u_2(t) = \int 3t\,dt = \frac{3}{2}t^2$$

$$\varphi_p = -t^3 e^{-2t} + \frac{3}{2}t^2\left(te^{-2t}\right)$$

$$= \frac{t^3}{2}e^{-2t}$$

All solutions: $\frac{t^3}{2}e^{-2t} + c_1 e^{-2t} + c_2 te^{-2t}$

$c_1, c_2 \in \mathbb{R}$.

- $\frac{d^2y}{dt^2} + y = \tan(t)$, $\quad -\frac{\pi}{2} < t < \frac{\pi}{2}$.

Associated homogeneous equation:

$$\frac{d^2y}{dt^2} + y = 0.$$

$$\chi(z) = z^2 + 1 = (z-i)(z+i)$$

Basis: $\{\cos(t), \sin(t)\}$

Take $\varphi_1 = \cos(t)$, $\varphi_2 = \sin(t)$.

$$W(\varphi_1, \varphi_2)(t) = \det\begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix} = 1.$$

$$u_1' = -\frac{\varphi_2 F}{W} = -\sin(t)\cdot\tan(t)$$

$$= -\frac{\sin^2(t)}{\cos(t)} = -\frac{1-\cos^2(t)}{\cos(t)}$$

$$= \frac{\cos^2(t)-1}{\cos(t)}$$

$$= \cos(t) - \sec(t)$$

$$u_1 = \int \cos(t) - \sec(t)\,dt$$

$$= \sin(t) - \underbrace{\ln(\sec(t) + \tan(t))}_{\int \sec(t)\,dt \text{ is quite tricky.}}$$

$$u_2' = \frac{\varphi_1 F}{W} = \frac{\cos(t)\tan(t)}{1} = \sin(t)$$

$$u_2 = \int \sin(t)\,dt = -\cos(t)$$

$$\varphi_p = \Big(\sin(t) - \ln(\sec(t) + \tan(t))\Big)\cos(t) + (-\cos(t))\sin(t)$$

$$= -\cos(t)\ln(\sec(t) + \tan(t))$$

All solutions:

$$-\cos(t)\ln\left(\sec(t)+\tan(t)\right)+c_1\cos(t)+c_2\sin(t),$$

$$c_1, c_2 \in \mathbb{R}.$$