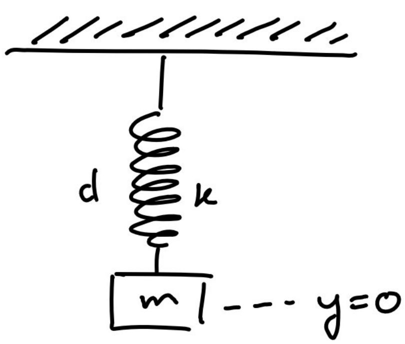
Mthe 237
Lecture 19
Oct. 24, 2017

Topic: Resonance in damped harmonic oscillators



To make the discussion more concrete, we shall concentrate on the spring-mass incarnation of a harmonic oscillator, but similar considerations hold for other types of oscillators.

We have seen a RLC circuit, a cork in a pond. There are many more.

Recall that the equation of motion is

$$\underbrace{m}_{\text{mass of object attached to spring}} \frac{d^2y}{dt^2} + \underbrace{d}_{\text{damping constant}} \frac{dy}{dt} + \underbrace{k}_{\text{spring constant}} y = \underbrace{F(t)}_{\text{driving force}}$$

Dividing by m and rewriting slightly,

$$\frac{d^2y}{dt^2} + 2\left(\frac{d}{2m}\right)\frac{dy}{dt} + \left(\frac{k}{m}\right)y = \frac{F}{m}$$

Introduce the notation

$$\frac{d}{2m} = \gamma ,$$

$$\frac{k}{m} = \omega_0^2 ,$$

"ω_0" is called the natural frequency of the system. It is the frequency the system would oscillate with without damping and forcing.

$$\frac{F}{m} = f .$$

Indeed, $\frac{d^2y}{dt^2} + \omega_0^2 y = 0$ has characteristic polynomial $\chi(z) = z^2 + \omega_0^2 = (z - i\omega_0)(z + i\omega_0)$. Hence solutions $c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t) = A\cos(\omega_0 t + \varphi)$

frequency of oscillation

With this notation, the equation becomes

$$\frac{d^2y}{dt^2} + 2\gamma \frac{dy}{dt} + \omega_0^2 y = f .$$

The associated homogeneous equation is

$$\frac{d^2y}{dt^2} + 2\gamma \frac{dy}{dt} + \omega_0^2 y = 0 .$$

This has characteristic polynomial

$$\chi(z) = z^2 + 2\gamma z + \omega_0^2 .$$

As before, the roots of the characteristic polynomial may be found using the quadratic formula:

$$\frac{-2\gamma \pm \sqrt{4\gamma^2 - 4\omega_0^2}}{2} = -\gamma \pm \sqrt{\gamma^2 - \omega_0^2}.$$

Suppose that $\gamma^2 - \omega_0^2 < 0$, so that the roots are complex.

A basis of solutions is then given by

$$\left\{ e^{-\gamma t} \cos\left(\sqrt{\omega_0^2 - \gamma^2}\, t\right),\ e^{-\gamma t} \sin\left(\sqrt{\omega_0^2 - \gamma^2}\, t\right) \right\}.$$

A solution

$$c_1 e^{-\gamma t} \cos\left(\sqrt{\omega_0^2 - \gamma^2}\, t\right) + c_2 e^{-\gamma t} \sin\left(\sqrt{\omega_0^2 - \gamma^2}\, t\right)$$

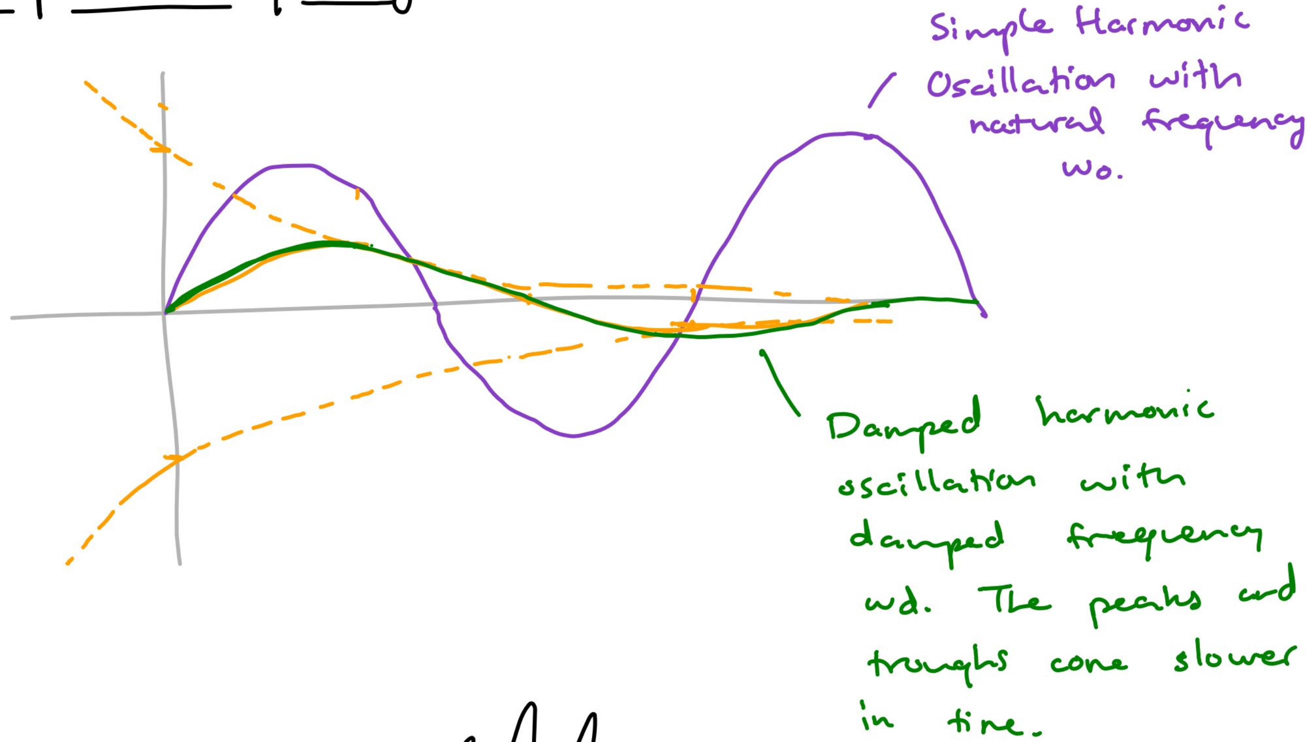
in the span of the basis may be written in phase-amplitude form as

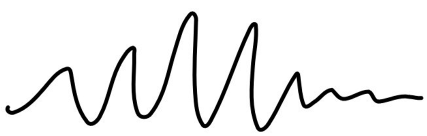
$$A e^{-\gamma t} \cos\left(\sqrt{\omega_0^2 - \gamma^2}\, t + \varphi\right), \quad \begin{array}{l} A \geq 0 \\ \varphi \in (-\pi, \pi]. \end{array}$$

We see that the frequency of the underdamped oscillation is $\sqrt{\omega_0^2 - \gamma^2} < \omega_0$.

It becomes equal to the natural frequency ω_0 when the damping γ is zero, as we may have expected.

We denote $\sqrt{\omega_0^2-\gamma^2}$ by ω_d, and call it the <u>damped frequency</u>.





(Relatively unimportant) aside: how to estimate how much smaller ω_d is than ω_0?

$$\frac{\omega_d}{\omega_0} = \frac{\sqrt{\omega_0^2-\gamma^2}}{\omega_0} = \sqrt{\frac{\omega_0^2-\gamma^2}{\omega_0^2}} = \sqrt{1-\frac{\gamma^2}{\omega_0^2}}$$

$=$ Expand in Taylor series.

A way of finding the Taylor expansion of $(1+x)^\alpha$, $\alpha \in \mathbb{R}$, without computing derivatives

makes use of:

Def. For $\alpha \in \mathbb{R}$ and $k \in \mathbb{N}$, the *generalized binomial coefficient* is defined to be

$$\binom{\alpha}{k} := \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!}$$

(Note when $\alpha \in \mathbb{Z}$, this is the same as the binomial coefficient defined previously:

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!} = \frac{1}{k!}\left(\frac{n(n-1)\cdots(n-k+1)(n-k)\cdots 2\cdot 1}{(n-k)(n-k-1)\cdots 2\cdot 1}\right)$$

$$= \frac{n(n-1)\cdots(n-k+1)}{k!}$$
)

Examples:

$$\binom{\frac{1}{2}}{0} = 1 \quad \text{(by convention)}$$

$$\binom{\frac{1}{2}}{1} = \frac{\frac{1}{2}}{1!} = \frac{1}{2}$$

$$\binom{\frac{1}{2}}{2} = \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!} = \frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2} = -\frac{1}{8}$$

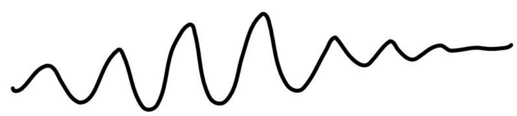
$$\binom{\frac{1}{2}}{3} = \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!} = \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{6} = \frac{3}{48} = \frac{1}{16}$$

We have Newton's binomial series:

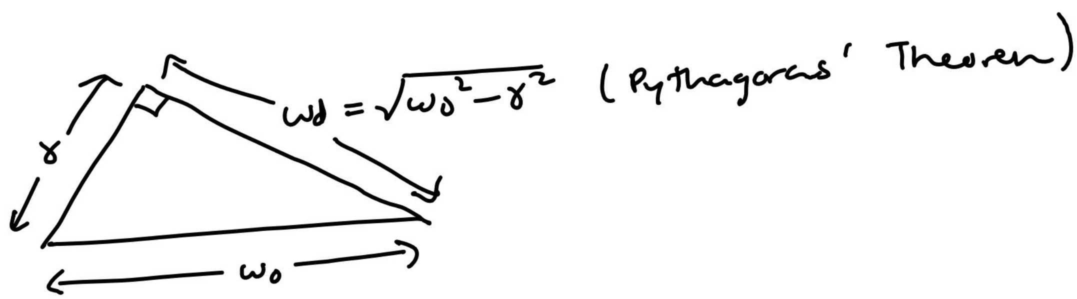
$$(1+x)^{\alpha} = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k.$$

Therefore, $$(1+x)^{1/2} = \sum_{k=0}^{\infty} \binom{\frac{1}{2}}{k} x^k = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + \cdots$$

Finally, $\sqrt{1-\frac{\gamma^2}{\omega_0^2}} = \left(1-\frac{\gamma^2}{\omega_0^2}\right)^{1/2} \approx 1 - \frac{1}{2}\frac{\gamma^2}{\omega_0^2} - \frac{1}{8}\frac{\gamma^4}{\omega_0^4} - \frac{1}{16}\frac{\gamma^6}{\omega_0^6}$



The relationship between the natural frequency ω_0, the damped frequency ω_d and γ may be displayed on a right triangle:



Now that we understand a bit more about the underdamped oscillations, including the damped frequency ω_d, let's study the following question:

What is the resulting motion if the damped spring-mass system is driven by a motor exerting the periodic force

$F(t) = m F_0 \cos(\omega t)$, with <u>driving</u> frequency ω?

(F_0 is a constant)

The equation of motion is

$$\frac{d^2y}{dt^2} + 2\gamma \frac{dy}{dt} + \omega_0^2 y = F_0 \cos(\omega t).$$

Let's use the method of undetermined coefficients to find a particular solution.

For $F_0 \cos(\omega t)$, a possible annihilator is

$$\left(\frac{d}{dt} - i\omega\right)\left(\frac{d}{dt} + i\omega\right).$$

Applying this to both sides of (*), we get the homogeneous linear equation

$$\left(\frac{d}{dt} - i\omega\right)\left(\frac{d}{dt} + i\omega\right)\left(\frac{d}{dt} - \left(-\gamma + \sqrt{\gamma^2 - \omega_0^2}\right)\right)\left(\frac{d}{dt} - \left(-\gamma - \sqrt{\gamma^2 - \omega_0^2}\right)\right) = 0$$

Because the roots of $\chi(z)$ have a real part (equal to $-\gamma$, which we are assuming is not zero), $\chi(\frac{d}{dt})$ has no shared factors with the annihilator. Therefore, discarding solutions to the associated homogeneous equation for now, we should look for φ_p in the span of

$$\underbrace{\{\cos(\omega t), \sin(\omega t)\}}_{\text{Basis elements corresponding to the annihilator.}}$$

$$\varphi_p(t) = c_1 \cos(\omega t) + c_2 \sin(\omega t)$$

$$\varphi_p'(t) = -\omega c_1 \sin(\omega t) + \omega c_2 \cos(\omega t)$$

$$\varphi_p''(t) = -\omega^2 c_1 \cos(\omega t) - \omega^2 c_2 \sin(\omega t)$$

Thus,

$$\varphi_p'' + 2\gamma \varphi_p' + \omega_0^2 \varphi_p$$

$$= \left(-\omega^2 c_1 + 2\gamma\omega c_2 + \omega_0^2 c_1\right) \cos(\omega t) + \left(-\omega^2 c_2 - 2\gamma\omega c_1 + \omega_0^2 c_2\right) \sin(\omega t)$$

For φ_p to be a particular solution, this should equal $F_0 \cos(\omega t)$ $(+\ 0 \cdot \sin(\omega t))$.

Matching coefficients, we obtain the system of equations

$$\left.\begin{array}{lr} (1) & (\omega_0^2 - \omega^2) c_1 + 2\gamma\omega c_2 = F_0 \\ (2) & (\omega_0^2 - \omega^2) c_2 - 2\gamma\omega c_1 = 0 \end{array}\right\}$$

From (2), $c_1 = \dfrac{\omega_0^2 - \omega^2}{2\gamma\omega} c_2$. Into (1),

$$\left(\frac{(\omega_0^2 - \omega^2)^2}{2\gamma\omega} + 2\gamma\omega \right) c_2 = F_0$$

So
$$\left(\frac{(\omega_0^2-\omega^2)^2+4\gamma^2\omega^2}{2\gamma\omega}\right) c_2 = F_0$$

$$\left.\begin{aligned} c_2 &= \left(\frac{2\gamma\omega}{(\omega_0^2-\omega^2)^2+4\gamma^2\omega^2}\right) F_0 \\ c_1 &= \left(\frac{\omega_0^2-\omega^2}{(\omega_0^2-\omega^2)^2+4\gamma^2\omega^2}\right) F_0 \end{aligned}\right\} \text{ are the solutions.}$$

We would like to write

$$c_1 \cos(\omega t) + c_2 \sin(\omega t) = A\cos(\omega t+\phi)$$

for some A and ϕ. Recall from the homework that

$$A\cos(\omega t+\phi) = A\left(\cos(\omega t)\cos(\varphi) - \sin(\omega t)\sin(\varphi)\right)$$
$$= \left(A\cos(\varphi)\right)\cos(\omega t) + \left(-A\sin(\varphi)\right)\sin(\omega t)$$

$$c_1 = A\cos(\varphi)$$
$$c_2 = -A\sin(\varphi)$$
(This uses uniqueness of a representation of a vector in terms of basis elements)

So $$c_1^2 + c_2^2 = A^2\left(\cos^2(\varphi)+\sin^2(\varphi)\right) = A^2.$$
$$A = \sqrt{c_1^2+c_2^2}$$

and $$-\frac{c_2}{c_1} = \tan(\varphi).$$

We compute, with the previously found c_1 and c_2, that

$$\frac{1}{F_0^2}(c_1^2 + c_2^2) = \left(\frac{(\omega_0^2 - \omega^2)}{(\omega_0^2 - \omega^2)^2 + 4\gamma^2\omega^2} \right)^2 + \left(\frac{2\gamma\omega}{(\omega_0^2 - \omega^2)^2 + 4\gamma^2\omega^2} \right)^2$$

$$= \frac{(\omega_0^2 - \omega^2)^2 + 4\gamma^2\omega^2}{\left[(\omega_0^2 - \omega^2)^2 + 4\gamma^2\omega^2\right]^2} = \frac{1}{(\omega_0^2 - \omega^2)^2 + 4\gamma^2\omega^2}$$

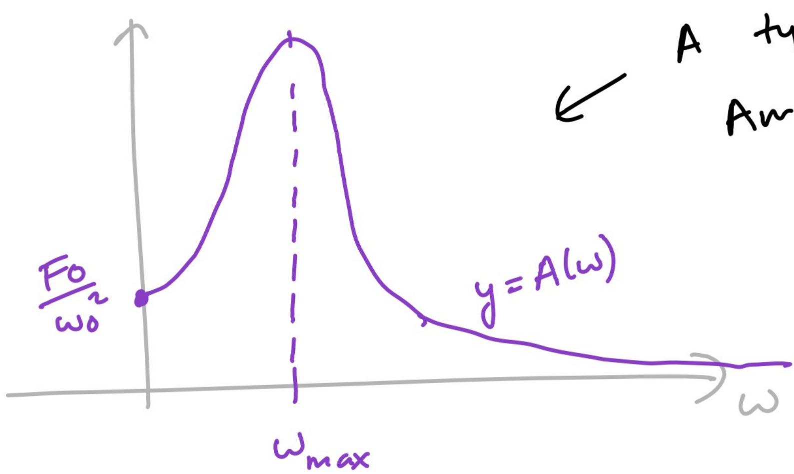
So that the amplitude of the particular solution φ_p is

$$A = \sqrt{c_1^2 + c_2^2} = \frac{F_0}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\gamma^2\omega^2}}$$

As $\omega \to 0$, $\quad A \to \dfrac{F_0}{\omega_0^2}$

$\omega \to \infty$, $\quad A \longrightarrow 0$

A typical graph of Amplitude as a function of ω



is called the resonant frequency

$\omega_{max} = \sqrt{\omega_0^2 - 2\gamma^2}$ (as you will check in homework!)

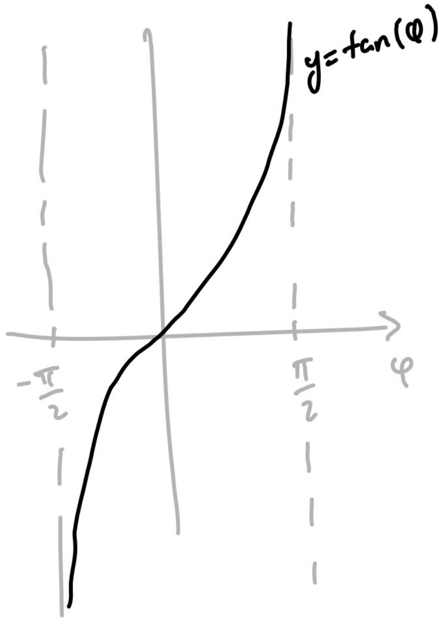
$$\frac{c_2}{c_1} = \frac{2\gamma\omega \Big/ (\omega_0^2-\omega^2)^2+4\gamma^2\omega^2}{(\omega_0^2-\omega^2) \Big/ (\omega_0^2-\omega^2)^2+4\gamma^2\omega^2} = \frac{2\gamma\omega}{\omega_0^2-\omega^2}$$

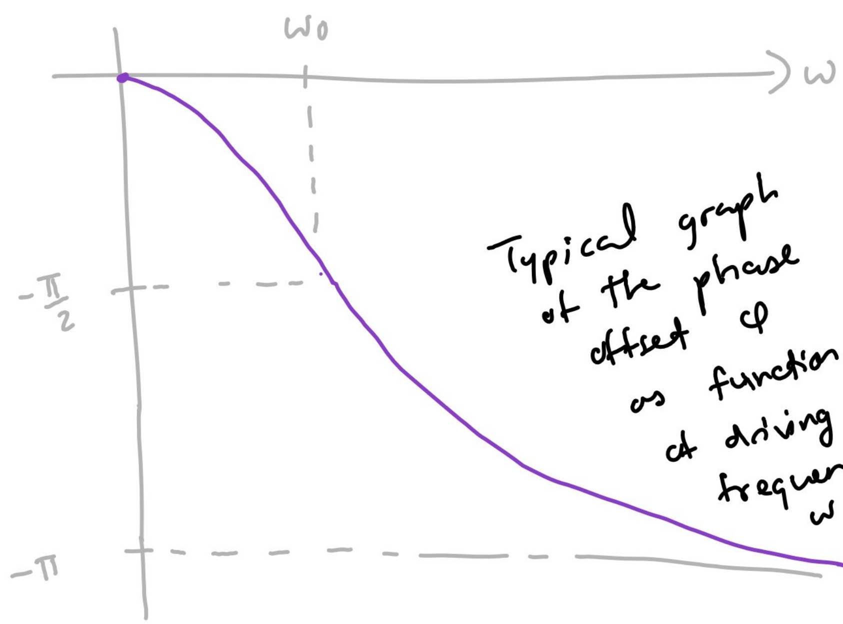
$$\tan\varphi = -\frac{c_2}{c_1} = \frac{2\gamma\omega}{\omega^2-\omega_0^2}$$

As $\omega \to 0$, $\tan\varphi \to 0$, so $\varphi \to 0$
The driving force and the motion are in phase

$\omega \to \omega_0$, $\tan\varphi \to \infty$, so $\varphi \to -\frac{\pi}{2}$
Natural frequency
The driving force and the motion are $\frac{\pi}{2}$ out of phase.

$\omega \to \infty$, $\tan\varphi \to 0$ again. this time $\varphi \to -\pi$





Typical graph of the phase offset φ as function of driving frequency ω.