Mthe 237
Lecture 18
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Topic: Method of Undetermined Coefficients

(The method is also called the annihilater method.)

Def. A differential operator $p(\frac{d}{dt})$ is said to be an <u>annihilator</u> of a function y over $I \subset \mathbb{R}$ if

$$p(\tfrac{d}{dt})y = 0 \quad \text{over } I.$$

The name comes from "nihil", which is a Latin word for "zero", so that "annihilator" means roughly "bringer to zero".

<u>Idea of the method</u>:

Suppose that we are trying to solve the linear equation with constant coefficients

$$(*) \quad \chi(\tfrac{d}{dt})y = F \quad \text{over } I,$$

and that we found $p_F(\frac{d}{dt})$, a polynomial differential operator with constant coefficients that annihilates F over I.

Applying $p_F(\frac{d}{dt})$ to both sides of $(*)$, we get

$$p_F(\tfrac{d}{dt})\left[\,\chi(\tfrac{d}{dt})y\,\right] = p_F(\tfrac{d}{dt})(F) = 0,$$

or, equivalently (by definition of operator product)

$$\left[\,p_F(\tfrac{d}{dt})\,\chi(\tfrac{d}{dt})\right]y = 0.$$

This is now a <u>homogeneous</u> linear differential equation with constant coefficients, and we have a general procedure for solving such equations.

If φ is a solution of $\chi(\frac{d}{dt})y = F$, then it is also a solution of $[p_F(\frac{d}{dt})\,\chi(\frac{d}{dt})]y = 0$,

$$\left\{\begin{matrix}\text{solutions of}\\ \chi(\tfrac{d}{dt})y = F\end{matrix}\right\} \subseteq \left\{\begin{matrix}\text{Solutions of}\\ [p_F(\tfrac{d}{dt})\,\chi(\tfrac{d}{dt})]y=0\end{matrix}\right\}.$$

Therefore, we can look for a particular solution φ_p of $\chi(\frac{d}{dt})y = F$ in the span of basis of solutions of $[p_F(\frac{d}{dt})\,\chi(\frac{d}{dt})]y = 0$: If $\{e_1, \dots, e_r\}$ is this basis, we can plug in

$$a_1e_1 + \dots + a_re_r \quad \text{into} \quad \chi(\tfrac{d}{dt})y = F,$$

and look for conditions on $a_1, \dots, a_r$ that make the linear combination a solution of $\chi(\frac{d}{dt})y = F$.

Example $\quad \dfrac{dy}{dt} + y = te^t.$

Here $F(t) = te^t$.

te^t is contained in the span of $\{e^t, te^t\}$,

which is the space of solutions of the diff. eq.

$$\left(\frac{d}{dt} - 1\right)^2 y = 0, \quad \text{or} \quad \frac{d^2y}{dt^2} - 2\frac{dy}{dt} + y = 0.$$

Therefore, we see that
$$p_F\left(\frac{d}{dt}\right) = \left(\frac{d}{dt} - 1\right)^2 = \left(\frac{d^2}{dt^2} - 2\frac{d}{dt} + 1\right)$$

is an annihilator of te^t.

The original differential equation is
$$\left(\frac{d}{dt} + 1\right) y = te^t.$$

Applying $p_F\left(\frac{d}{dt}\right) = \left(\frac{d}{dt} - 1\right)^2$ to both sides,

$$\left(\frac{d}{dt} - 1\right)^2 \left(\frac{d}{dt} + 1\right) y = \left(\frac{d}{dt} - 1\right)^2 (te^t) = 0.$$

The basis of solutions of

$$\left(\frac{d}{dt} - 1\right)^2 \left(\frac{d}{dt} + 1\right) y = 0$$

is given by $\{ \underbrace{e^t, te^t}_{\left(\frac{d}{dt} - 1\right)^2}, \underbrace{e^{-t}}_{\left(\frac{d}{dt} + 1\right)} \}$

Look for a particular solution of $\left(\frac{d}{dt}+1\right)y = te^t$ in the span of e^t, te^t and e^{-t}.

[Since e^{-t} is a solution of the associated homogeneous equation $\left(\frac{d}{dt}+1\right)y=0$, we can disregard it for purposes of finding a solution of $\left(\frac{d}{dt}+1\right)y = te^t$, saving ourselves some work.]

$$\varphi_p(t) = a_1 e^t + a_2 te^t, \qquad a_1, a_2 \in \mathbb{R}$$

$$\frac{d\varphi_p}{dt}(t) = a_1 e^t + a_2 \underbrace{\left(e^t + te^t\right)}_{\text{product rule}}$$

$$= (a_1 + a_2)e^t + a_2 te^t.$$

Thus, $\frac{d\varphi_p}{dt} + \varphi_p =$

$$\left((a_1+a_2)e^t + a_2 te^t\right) + \left(a_1 e^t + a_2 te^t\right)$$

$$= (2a_1 + a_2)e^t + 2a_2 te^t$$

We want this to equal $te^t = 0 \cdot e^t + 1 \cdot te^t$,

so

$$\left.\begin{aligned} (1) \quad & 2a_1 + a_2 = 0 \\ (2) \quad & 2a_2 = 1 \end{aligned}\right\}$$

From (2), $a_2 = \frac{1}{2}$. Thus, in (1), $2a_1 + \frac{1}{2} = 0$

$$a_1 = -\frac{1}{4}.$$

So, $\varphi_p(t) = -\frac{1}{4}e^t + \frac{1}{2}te^t$

is a particular solution of $\frac{dy}{dt} + y = te^t$.

$\Bigg($ We can check this:

$$\frac{d\varphi_p}{dt} = -\frac{1}{4}e^t + \frac{1}{2}(e^t + te^t)$$

$$\frac{d\varphi_p}{dt} + \varphi_p = -\frac{1}{4}e^t + \frac{1}{2}e^t + \frac{1}{2}te^t - \frac{1}{4}e^t + \frac{1}{2}te^t$$

$$= te^t. \Bigg)$$

Last time, we have seen that the set of solutions of $\frac{dy}{dt} + y = te^t$ is equal to

$\varphi_p + \mathrm{Sol}\left(\frac{dy}{dt} + y = 0\right)$, where φ_p is any solution of $\frac{dy}{dt} + y = te^t$.

Since $\mathrm{Sol}\left(\frac{dy}{dt}+y=0\right) = \mathrm{span}\{e^{-t}\}$,

we see that the set of solutions of

$\frac{dy}{dt}+y = te^t$ is equal to

$$\underbrace{-\frac{1}{4}e^t + \frac{1}{2}te^t}_{\varphi_p(t)} + \underbrace{a e^{-t}}_{\mathrm{Sol}\left(\frac{dy}{dt}+y=0\right)}, \quad a \in \mathbb{R}$$

The constant a may be found by imposing an initial condition, as usual.

<u>Example</u> $\frac{dy}{dt} - y = te^t$.

As in the previous example, we can take

$$p_F\left(\frac{d}{dt}\right) = \left(\frac{d}{dt}-1\right)^2$$

as an annihilator of te^t.

Applying $p_F\left(\frac{d}{dt}\right)$ to $\left(\frac{d}{dt}-1\right)y = te^t$, we get

$$\left(\frac{d}{dt}-1\right)^2\left(\frac{d}{dt}-1\right)y = 0, \quad \text{or}$$

$$\left(\frac{d}{dt}-1\right)^3 y = 0$$

Basis of solutions: $\{e^t, te^t, t^2e^t\}$

Look for a solution to $\frac{dy}{dt} - y = te^t$ in the span of this basis. As before, e^t is a solution to $\frac{dy}{dt} - y = 0$ and may be disregarded in looking for a particular solution.

$$\varphi_p(t) = a_1 te^t + a_2 t^2 e^t.$$

$$\frac{d\varphi_p}{dt}(t) = a_1(e^t + te^t) + a_2(2te^t + t^2e^t)$$

$$= a_1 e^t + (a_1 + 2a_2)te^t + a_2 t^2 e^t$$

$$\frac{d\varphi_p}{dt} - \varphi_p = (a_1 - 0)e^t + ((a_1 + 2a_2) - a_1)te^t + \cdots$$

$$\cdots + (a_2 - a_2)t^2e^t$$

$$= a_1 e^t + 2a_2 te^t$$

Want this to be equal to

$$te^t = 0 \cdot e^t + 1 \cdot te^t, \text{ so}$$

$$\left.\begin{aligned} a_1 &= 0 \\ 2a_2 &= 1 \end{aligned}\right\} \Rightarrow \quad a_1 = 0, \quad a_2 = \frac{1}{2}.$$

We have found the particular solution

$$\varphi_p(t) = \frac{1}{2}t^2 e^t$$

$$\left(\text{Check:} \quad \frac{d\varphi_p}{dt} = te^t + \frac{1}{2}t^2e^t, \quad \frac{d\varphi_p}{dt} - \varphi_p = te^t + \frac{1}{2}t^2e^t - \frac{1}{2}t^2e^t = te^t \right)$$

All solutions of $\frac{dy}{dt} - y = te^t$ lie in the affine subspace

$$\varphi_p + \mathrm{Sol}\left(\frac{dy}{dt} - y = 0\right),$$

so every solution has the form

$$\frac{1}{2}t^2e^t + a\,e^t, \qquad a \in \mathbb{R}.$$

Remark. An interesting feature of the second example is that, due to the shared root of $P_F(\frac{d}{dt})$ and $\chi(\frac{d}{dt})$, the "candidate solution" changed from

$$a_1 e^t + a_2 te^t \quad \text{to}$$

$$a_1 te^t + a_2 t^2 e^t.$$

§ Quasipolynomials

Q: For which functions can we find an annihilator $p_F(\frac{d}{dt})$ that is a polynomial differential operator with constant coefficients?

A: Such functions are solutions to the differential equation

$$\underbrace{p_F(\tfrac{d}{dt})y = 0}$$

Homogeneous Linear Diff. Eq. with constant coefficients

From our understanding of what solutions to such equations look like, we define:

Def. A quasipolynomial is a function of one of the following forms:

- (Polynomial in t) e^{rt}, $r \in \mathbb{R}$
- (Polynomial in t) $e^{\sigma t} \cos(\omega t)$, $\sigma, \omega \in \mathbb{R}$
- (Polynomial in t) $e^{\sigma t} \sin(\omega t)$, $\sigma, \omega \in \mathbb{R}$

or a sum of such functions.

We can now summarize the above discussion as follows:

Prop. A function y has an annihilator that is polynomial with constant coefficients if and only if y is a quasipolynomial.

Examples of finding $p_F(\frac{d}{dt})$:

- $$(1 + 5t + 17t^3)e^{5t} \in \underbrace{\text{span}\{e^{5t}, te^{5t}, \dots, t^3e^{5t}\}}_{\substack{\text{Space of solutions of} \\ (\frac{d}{dt}-5)^4 y = 0}}$$

$$p_F(\tfrac{d}{dt}) = (\tfrac{d}{dt} - 5)^4 \text{ works.}$$

- $$\cos(wt) \in \underbrace{\text{span}\{\cos(wt), \sin(wt)\}}_{\substack{\text{space of solutions of} \\ (\frac{d}{dt} - iw)(\frac{d}{dt} + iw) y = 0}}$$

$$p_F(\tfrac{d}{dt}) = (\tfrac{d}{dt} - iw)(\tfrac{d}{dt} + iw) \text{ works.}$$

- $t^2 e^{5t} \sin(7t)$ ...
$$p_F(\tfrac{d}{dt}) = \left[\left(\tfrac{d}{dt} - (5+7i)\right)\left(\tfrac{d}{dt} - (5-7i)\right)\right]^3$$

works.

Method of undetermined coefficients:

To solve $\underbrace{\chi(\tfrac{d}{dt})}_{\text{constant coefficients}} y = \underbrace{q}_{\text{quasipolynomial}}$

1. Find an annihilator $p_q(\frac{d}{dt})$ of q.

2. Find basis of solutions of homogeneous linear eq.

$$p_q(\tfrac{d}{dt})\,\chi(\tfrac{d}{dt})\,y = 0.$$

3. Find particular solution φ_p of $\chi(\frac{d}{dt})y = q$ in span of basis elements from step 2.

   [Basis elements that are solutions of $\chi(\frac{d}{dt})y = 0$ may be disregarded to save some work.]

4. Find basis of solutions of $\chi(\frac{d}{dt})y = 0$.

Set of all solutions of $\chi(\frac{d}{dt})y = q$ is equal to

$$\varphi_p + \mathrm{Sol}\left(\chi(\tfrac{d}{dt})y = 0\right).$$