Mthe 237
Lecture 17
Oct. 18, 2017

Topic: General linear equations — Basic Theory

A linear differential equation is one of the form

$$(L) \quad \frac{d^r y}{dt^r} + a_{r-1}(t)\frac{d^{r-1}y}{dt^{r-1}} + \cdots a_1(t)\frac{dy}{dt} + a_0(t)y = F(t) \quad (t \in I)$$

When we want to specify that the function F on the right-hand side is not equal to 0 for all $t \in I$, we say that (L) is nonhomogeneous.

Similarly to the linear homogeneous case, we have the basic

Theorem [Existence and Uniqueness for Linear Equations]

If the functions $a_{r-1}, \ldots, a_1, a_0, F$ in (L) are continuous over an open interval I, then there exists a unique solution $\varphi: I \longrightarrow \mathbb{R}$ of (L) satisfying the initial conditions

$$\varphi(t_0) = \varphi_0^{(0)}, \quad \frac{d\varphi}{dt}(t_0) = \varphi_0^{(1)}, \quad \ldots, \quad \frac{d^{r-1}\varphi}{dt^{r-1}}(t_0) = \varphi_0^{(r-1)}$$

$$t_0 \in I, \quad \varphi_0^{(0)}, \ldots, \varphi_0^{(r-1)} \in \mathbb{R}$$

Because of the term $F(t)$, solutions to (L) no longer form a vector space (unless $F=0$). However, the set of solutions does have some structure.

The linear homogeneous equation

Def.

$$\frac{d^r y}{dt^r} + a_{r-1}(t)\frac{d^{r-1}y}{dt^{r-1}} + \cdots + a_1(t)\frac{dy}{dt} + a_0(t)y = 0$$

obtained by setting the right side of (L) to 0 is called the homogeneous equation associated to (L). We denote it by (H_L).

Theorem. Let φ_p be a fixed solution of (L) over $I \subseteq \mathbb{R}$. The set of solutions of (L) is equal to

$$\varphi_p + \mathrm{Sol}(H_L) \overset{\text{def.}}{=} \left\{ \varphi_p + \varphi_h : \begin{array}{l} \varphi_h \text{ is a solution} \\ \text{of } (H_L) \text{ over } I \end{array} \right\}$$

(The addition is taking place in $C^r(I,\mathbb{R})$).

Remark:

In the proof of this theorem, we work with

$$\chi\left(\frac{d}{dt}\right) = \frac{d^r}{dt^r} + a_{r-1}(t)\frac{d^{r-1}}{dt^{r-1}} + \cdots + a_1(t)\frac{d}{dt} + a_0(t).$$

This is a polynomial differential operator with time-varying coefficients.

$\chi(\frac{d}{dt})$ is still linear,

$$\chi(\tfrac{d}{dt})(c_1\varphi_1 + c_2\varphi_2) = c_1\,\chi(\tfrac{d}{dt})(\varphi_1) + c_2\,\chi(\tfrac{d}{dt})(\varphi_2)$$

for $c_1, c_2 \in \mathbb{R}$
$\varphi_1, \varphi_2 : I \to \mathbb{R}$

but we must be careful not to assume that such operators are commutative (this is false in general, as we have seen before).

Proof of Theorem. Using the operator above, the differential equations may be written as

$$(L) \qquad \chi(\tfrac{d}{dt})y = F,$$

$$(H_L) \qquad \chi(\tfrac{d}{dt})y = 0.$$

Suppose that γ is a solution of (L) over I. By linearity of $\chi(\frac{d}{dt})$, we have

$$\chi(\tfrac{d}{dt})(\gamma - \varphi_p) = \chi(\tfrac{d}{dt})(\gamma) - \chi(\tfrac{d}{dt})(\varphi_p)$$
$$= F - F = 0$$

Therefore, the function $(\gamma - \varphi_p)$ is a solution of the associated linear equation (H_L).

$\gamma - \varphi_p \in \mathrm{Sol}(H_L)$, so $\gamma \in \varphi_p + \mathrm{Sol}(H_L)$.

This shows the inclusion of sets

$$\left\{\begin{matrix}\text{solutions of}\\ (L)\end{matrix}\right\} \subseteq \varphi_p + \mathrm{Sol}(H_L).$$

Conversely, suppose that $\varphi_h \in \mathrm{Sol}(H_L)$. Then

$$\begin{aligned}\chi\left(\tfrac{d}{dt}\right)(\varphi_p+\varphi_h) &= \chi\left(\tfrac{d}{dt}\right)(\varphi_p) + \chi\left(\tfrac{d}{dt}\right)(\varphi_h)\\ &= F + 0 = F,\end{aligned}$$

so that $(\varphi_p+\varphi_h)$ is a solution of (L).

This shows the inclusion of sets

$$\left\{\begin{matrix}\text{Solutions of}\\ (L)\end{matrix}\right\} \supseteq \varphi_p + \mathrm{Sol}(H_L).$$

Therefore, combining the two inclusions shown above,

$$\left\{\begin{matrix}\text{Solutions of}\\ (L)\end{matrix}\right\} = \varphi_p + \mathrm{Sol}(H_L). \qquad \blacksquare$$

STRATEGY FOR LINEAR EQUATIONS:

1. Find a <u>particular solution</u> φ_p of (L) (this is <u>any</u> fixed solution of (L)).
2. Solve the associated homogeneous equation (H_L).
3. Form $\varphi_p + \mathrm{Sol}(H_L)$ — this is the set of all solutions of (L).

§ Geometry of {Solutions of (L)}:

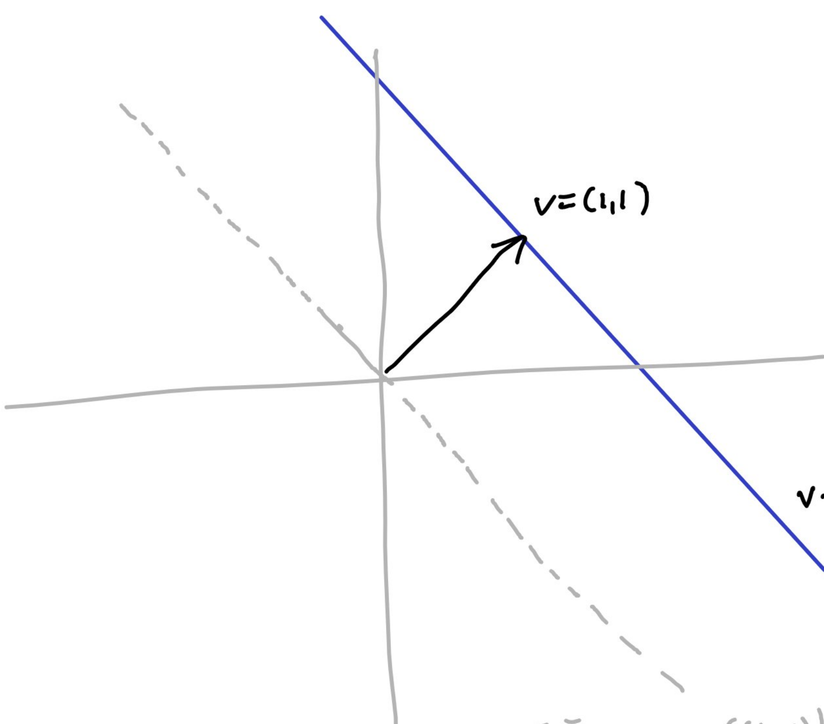
Reminder: Let V be a vector space
$v \in V$ a vector
$W \subseteq V$ a linear subspace

The set of vectors
$$v + W = \{ v + w : w \in W \}$$
is called an affine subspace of V.

(We can think of $v + W$ as a translation of every vector in W by the vector v.)

Examples:

- In $V = \mathbb{R}^2$, $v = (1,1)$, $W = \text{span}((1,-1))$

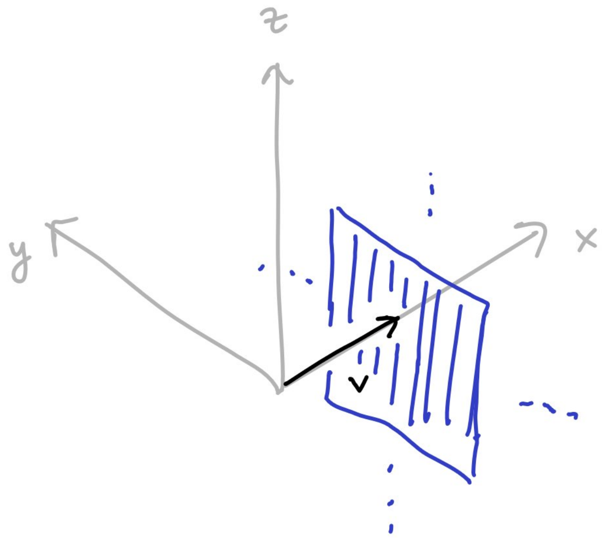


$$v + W = \{ (1,1) + c(1,-1) : c \in \mathbb{R} \}$$
$$= \{ (1+c, 1-c) : c \in \mathbb{R} \}$$

$W = \text{span}((1,-1))$

• In $V = \mathbb{R}^3$, $\quad v = (1,0,0) \quad W = \mathrm{span}((0,1,0),(0,0,1))$

(the yz-plane)



The theorem that the set of solutions of (L) is equal to

$$\varphi_p + \mathrm{Sol}(H_L)$$

may be viewed as saying that the set of solutions of (L) is an affine subspace of $C^r(I,\mathbb{R})$.

It again has dimension equal to the order of (L), so the family of solutions depends on $r = \mathrm{order}(L)$ parameters.