Mthe 237
Lecture 17
Oct. 18, 2017

Topic: · End of Proof of linear independence

Last time:

Lemma $I \neq \emptyset$, open

$w_1, \ldots, w_k \in \mathbb{C}$ pairwise distinct

$p_1(t), \ldots, p_k(t)$ polynomials with complex coefficients

If $p_1(t)e^{w_1 t} + \cdots + p_k(t)e^{w_k t} = 0$ for all $t \in I$,

then $p_1(t) = \cdots = p_k(t) = 0$ for all $t \in I$,

so that p_j is the zero polynomial for $j = 1, \ldots, k$.

Suppose that we have a homogeneous linear equation with characteristic polynomial

$$\chi(z) = (z-r_1)^{m_1} \cdots (z-r_k)^{m_k}\left[(z-z_1)(z-\bar{z}_1)\right]^{n_1} \cdots \left[(z-z_\ell)(z-\bar{z}_\ell)\right]^{n_\ell}$$

Suppose

$$a_{1,0}e^{r_1 t} + \cdots + a_{1,m_1-1}t^{m_1-1}e^{r_1 t} + \cdots$$

$$\cdots + a_{k,0}e^{r_k t} + \cdots + a_{k,m_k-1}t^{m_k-1}e^{r_k t} + \cdots$$

$$\cdots + b_{1,0}e^{z_1 t} + \cdots + b_{1,n_1-1}t^{n_1-1}e^{z_1 t} + \cdots$$

$$\cdots + c_{1,0}e^{\bar{z}_1 t} + \cdots + c_{1,n_1-1}t^{n_1-1}e^{\bar{z}_1 t} + \cdots$$

with a's, b's, c's in $\mathbb{C}$

$$\cdots + b_{\ell,0} e^{z_\ell t} + \cdots + b_{\ell,n_\ell-1} t^{n_\ell-1} e^{z_\ell t} + \cdots$$

$$\cdots + c_{\ell,0} e^{\bar{z}_\ell t} + \cdots + c_{\ell,n_\ell-1} t^{n_\ell-1} e^{\bar{z}_\ell t} = 0.$$

Can rewrite as

$$\left(a_{1,0} + a_{1,1} t + \cdots + a_{1,m_1-1} t^{m_1-1}\right) e^{r_1 t} + \cdots$$

$$+ \cdots$$

$$+ \left(a_{k,0} + a_{k,1} t + \cdots + a_{k,m_k-1} t^{m_k-1}\right) e^{r_k t} +$$

$$+ \left(\underbrace{b_{1,0} + b_{1,1} t + \cdots + b_{1,n_1-1} t^{n_1-1}}_{\text{Polynomial with complex coefficients}}\right) e^{z_1 t} +$$

$$\vdots$$

$$+ \left(c_{\ell,0} + c_{\ell,1} t + \cdots + c_{\ell,n_\ell-1} t^{n_\ell-1}\right) e^{\bar{z}_\ell t} = 0$$

We are in the situation of the lemma.

Conclude that

$$\left(a_{1,0} + a_{1,1} t + \cdots + a_{1,m_1-1} t^{m_1-1}\right) = 0 \quad \text{for all } t \in I,$$

$$\vdots$$

$$\left(c_{\ell,0} + c_{\ell,1} t + \cdots + c_{\ell,n_\ell-1} t^{n_\ell-1}\right) = 0 \quad \text{for all } t \in I$$

Hence $a_{1,0} = a_{1,1} = \cdots = c_{\ell,n_\ell-1} = 0.$

The set

$$\{e^{r_1 t}, \dots, t^{m_u - 1} e^{r_u t}, e^{z_1 t}, \dots, t^{n_e - 1} e^{z_e t}, e^{\bar{z}_1 t}, \dots, t^{n_e - 1} e^{z_e t}\}$$

is a linearly independent subset of

$$C^r(I, \mathbb{C})$$

<u>Lemma</u>. Suppose $\{v_1, v_2, v_3, \dots, v_r\}$ is a linearly independent subset of a complex vector space V. Then the set $\left\{\frac{v_1 + v_2}{2}, \frac{v_1 - v_2}{2i}, v_3, \dots, v_r\right\}$ is again linearly independent.

<u>Argument 1</u>:

$$\begin{pmatrix} \frac{v_1+v_2}{2} \\ \frac{v_1-v_2}{2i} \\ v_3 \\ \vdots \\ v_r \end{pmatrix} = \left(\begin{array}{cc|cccc} \frac{1}{2} & \frac{1}{2} & & & & \\ \frac{1}{2i} & -\frac{1}{2i} & & & 0 & \\ \hline & & 1 & & & \\ & & & 1 & & \\ & 0 & & & \ddots & \\ & & & & & 1 \end{array}\right) \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_r \end{pmatrix}.$$

$\downarrow$

det of this matrix is

$$\det\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2i} & -\frac{1}{2i} \end{pmatrix} = -\frac{1}{4i} - \frac{1}{4i} = -\frac{1}{2i} \neq 0.$$

Invertible linear transformations send linearly independent sets to linearly independent sets.

$\blacksquare$

<u>Argument 2</u>: Suppose that

$$a_1\left(\frac{v_1+v_2}{2}\right) + a_2\left(\frac{v_1-v_2}{2i}\right) + a_3 v_3 + \cdots + a_r v_r = 0.$$

$$\left(\frac{a_1 + a_2/i}{2}\right) v_1 + \left(\frac{a_1 - a_2/i}{2}\right) v_2 + a_3 v_3 + \cdots + a_r v_r = 0.$$

$$\left.\begin{aligned} \frac{a_1 + \frac{a_2}{i}}{2} &= 0 \\ \frac{a_1 - \frac{a_2}{i}}{2} &= 0 \end{aligned}\right\} \text{ Solve to show } a_1 = a_2 = 0.$$

and $a_3 = a_4 = \cdots = a_r = 0$ $\blacksquare$

Now apply lemma repeatedly to transform the set

$$\{e^{r_1 t}, \ldots, t^{m_k - 1} e^{r_k t}, e^{z_1 t}, \ldots, t^{n_\ell - 1} e^{z_\ell t}, e^{\bar{z}_1 t}, \ldots, t^{n_\ell - 1} e^{\bar{z}_\ell t}\}$$

to

$$\{e^{r_1 t}, \ldots, t^{m_k - 1} e^{r_k t}, e^{\sigma_1 t}\cos(\omega_1 t), e^{\sigma_1 t}\sin(\omega_1 t), \ldots,$$
$$t^{n_1 - 1}\cos(\omega_1 t), t^{n_1 - 1}\sin(\omega_1 t), \ldots, t^{n_\ell - 1}\sin(\omega_\ell t)\}$$

For example, taking $v_1 = t^2 e^{z_1 t}$ and $v_2 = t^2 e^{\bar{z}_1 t}$, with $z_1 = \sigma_1 + i\omega_1$,

$$\frac{v_1+v_2}{2} = t^2\left(\frac{e^{z_1 t}+e^{\bar{z}_1 t}}{2}\right) = t^2 e^{\sigma_1 t}\left(\frac{e^{i\omega_1 t}+e^{-i\omega_1 t}}{2}\right)$$

$$= t^2 e^{\sigma_1 t}\cos(\omega_1 t) \quad \text{and}$$

$$\frac{v_1-v_2}{2i} = t^2\left(\frac{e^{z_1 t}-e^{\bar{z}_1 t}}{2i}\right) = t^2 e^{\sigma_1 t}\left(\frac{e^{i\omega_1 t}-e^{-i\omega_1 t}}{2i}\right)$$

$$= t^2 e^{\sigma_1 t}\sin(\omega_1 t)$$

So,

$$\{e^{r_1 t}, \ldots, t^{m_k - 1} e^{r_k t}, e^{\sigma_1 t}\cos(\omega_1 t), e^{\sigma_1 t}\sin(\omega_1 t), \ldots, t^{n_1 - 1}\cos(\omega_1 t), t^{n_1 - 1}\sin(\omega_1 t), \ldots, t^{n_e - 1}\sin(\omega_e t)\}$$

is a linearly independent subset of $C^r(I, \mathbb{C})$ (regarded as a complex vector space).

However, each element is actually in $C^r(I, \mathbb{R})$.

If they were linearly dependent in $C^r(I, \mathbb{R})$, the same linear relation would hold in $C^r(I, \mathbb{C})$. Since the latter is impossible, we conclude that the list of functions produced by the procedure for solving homogeneous linear equations with constant coefficients is indeed linearly independent, hence a <u>basis</u>.