MTHE 237
Lecture 16
Oct. 17, 2017

Topics: Lemma toward proving linear independence of solutions produced by procedure for solving Linear Homogeneous Differential Equations with Constant Coefficients

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§ Higher-Order Product Rule

Q: What is $\frac{d^r}{dt^r}\left(f(t)\,g(t)\right)$?

A: **Theorem.** For $r \geq 0$, and f and $g \in C^r(I, \mathbb{R})$,

$$\frac{d^r}{dt^r}\left(f(t)g(t)\right) = \binom{r}{0}\frac{d^r f}{dt^r}g + \binom{r}{1}\frac{d^{r-1}f}{dt^{r-1}}\frac{dg}{dt} + \cdots$$
$$\cdots + \binom{r}{r-1}\frac{df}{dt}\frac{d^{r-1}g}{dt^{r-1}} + \binom{r}{r} f \frac{d^r g}{dt^r}.$$

What do the symbols in the statement mean?

Def. For a nonnegative integer n, the integer $n!$, called n factorial, is defined recursively as

$$0! = 1,$$
$$n! = n(n-1)!$$

Examples: $0! = 1$, $1! = 1$, $2! = 2 \cdot 1$, $3! = 3 \cdot 2 \cdot 1 = 6$,

$4! = 24$, $5! = 120$, $6! = 720$, $7! = 5040$, ...

Def. For a pair of integers n and k, define

$$\binom{n}{k} = \begin{cases} \dfrac{n!}{k!(n-k)!}, & 0 \le k \le n, \\ 0, & \text{otherwise}. \end{cases}$$

Examples: $\binom{6}{4} = \dfrac{6!}{4!(6-4)!} = \dfrac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{(4 \cdot 3 \cdot 2 \cdot 1)(2 \cdot 1)} = 15,$

$$\binom{5}{1} = \frac{5!}{1!(5-1)!} = \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{(1)(4 \cdot 3 \cdot 2 \cdot 1)} = 5.$$

The symbol $\binom{n}{k}$ is pronounced "n choose k" and is called a binomial coefficient.

The reason for the name is the binomial formula:

For any $n \ge 0$,

$$(x+y)^n = \binom{n}{0} x^n y^0 + \binom{n}{1} x^{n-1} y + \cdots + \binom{n}{n-1} x\, y^{n-1} + \binom{n}{n} x^0 y^n.$$

$$= \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k.$$

Aside: The reason for "n choose k" is that $\binom{n}{k}$ is the number of k-element subsets of a n-element set.

$$\frac{\left(\begin{matrix} n \text{ choices for} \\ \text{first element} \end{matrix}\right)\left(\begin{matrix} n-1 \text{ choices for} \\ \text{second element} \end{matrix}\right) \cdots \left(\begin{matrix} n-k+1 \text{ choices} \\ \text{for } k^{th} \text{ element} \end{matrix}\right)}{k! \text{ rearrangements}}$$

$$= \frac{n(n-1)\cdots(n-k+1)}{k!}\,\frac{(n-k)(n-k-1)\cdots 2\cdot 1}{(n-k)(n-k-1)\cdots 2\cdot 1} = \frac{n!}{k!\,(n-k)!}$$

$\binom{n}{k}$ for low n and k are conveniently computed using Pascal's triangle:

$$\begin{array}{ccccccc}
 & & & \binom{0}{0} & & & \\
 & & \binom{1}{0} & & \binom{1}{1} & & \\
 & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & \\
\binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} \\
 & & & \vdots & & &
\end{array}
\qquad
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 1 & & 1 & & \\
 & 1 & & 2 & & 1 & \\
1 & & 3 & & 3 & & 1
\end{array}$$

To find entry, add the one immediately above and to the left and the one immediately above and to the right.

This is reflected in the identity

$$\binom{n-1}{k-1} + \binom{n-1}{k} = \binom{n}{k}$$

e.g. $$\underset{\substack{\shortparallel \\ 1}}{\binom{2}{0}} + \underset{\substack{\shortparallel \\ 2}}{\binom{2}{1}} = \underset{\substack{\shortparallel \\ 3}}{\binom{3}{1}}$$

Examples of product rules for $r=0,1,2,3$:

$$\frac{d^0}{dt^0}(fg) = \binom{0}{0} fg = fg \,;$$

$$\frac{d}{dt}(fg) = \binom{1}{0}\frac{df}{dt} g + \binom{1}{1} f \frac{dg}{dt}$$

$$= \frac{df}{dt} g + f\frac{dg}{dt} \quad (\text{Usual product rule});$$

$$\frac{d^2}{dt^2}(fg) = \binom{2}{0}\frac{d^2f}{dt^2} g + \binom{2}{1}\frac{df}{dt}\frac{dg}{dt} + \binom{2}{2} f \frac{d^2g}{dt^2}$$

$$= \frac{d^2f}{dt^2} g + 2\frac{df}{dt}\frac{dg}{dt} + f\frac{d^2g}{dt^2};$$

$$\frac{d^3}{dt^3}(fg) = \frac{d^3f}{dt^3} g + 3\frac{d^2f}{dt^2}\frac{dg}{dt} + 3\frac{df}{dt}\frac{d^2g}{dt^2} + f\frac{d^3g}{dt^3};$$

$$(\dots)$$

You are encouraged to prove the higher-order product rule by induction on r, the usual product rule, and the identity $\binom{n-1}{k-1} + \binom{n-1}{k} = \binom{n}{k}$.

§ Fact about polynomials with complex coefficients:

Let $p(z) = a_r z^r + a_{r-1} z^{r-1} + \dots + a_1 z + a_0$, $a_k \in \mathbb{C}$.

If there exist $w_1, \dots, w_{r+1}$ with $p(w_k) = 0$ for each k, then p is the zero polynomial and

$a_r = a_{r-1} = \dots = a_1 = a_0 = 0.$

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We are now ready to begin the proof that the sets of solutions produced by our procedure for solving linear homogeneous differential equations with constant coefficients are linearly independent.

Lemma. Let I be a nonempty open interval in $\mathbb{R}$.

$w_1, \dots, w_k \in \mathbb{C}$ pairwise distinct

$p_1(t), \dots, p_k(t)$, $t \in I$, polynomials with complex coefficients.

If $p_1(t)e^{w_1 t} + \dots + p_k(t)e^{w_k t} = 0$ for all $t \in I$,

then $p_1(t) = p_2(t) = \dots = p_k(t) = 0$ for all $t \in I$, so

that $p_1, p_2, \dots, p_k$ are zero polynomials.

Proof: Proceed by induction on k.

For $k=1$: $p_1(t)e^{w_1 t} = 0$ for all $t \in I$

Implies that $p_1(t) = 0$ for all $t \in I$ upon dividing by $e^{w_1 t}$. Claim is checked for $k=1$.

Suppose claim is true for $k-1$.

Suppose $p_1(t)e^{w_1 t} + \dots + p_k(t)e^{w_k t} = 0$ for all $t \in I$.

Dividing by $e^{w_k t}$, get

$$p_1(t)e^{(w_1-w_k)t} + \cdots + p_{k-1}(t)e^{(w_{k-1}-w_k)t} + p_k(t) = 0$$

for all $t \in I$.

Differentiate this expression $r \geq \operatorname{ord}(p_k)+1$ times.

Note $\frac{d^r}{dt^r} p_k(t) = 0$ with this choice of r;

we are setting up for induction.

$$\frac{d^r}{dt^r}\left(p_1(t)e^{(w_1-w_k)t}\right)$$

$$= \sum_{j=0}^{r} \binom{r}{j} \frac{d^{r-j}p_1}{dt^{r-j}}(t)\,(w_1-w_k)^j e^{(w_1-w_k)t}$$

$$= \left[\sum_{=0}^{r} \binom{r}{j} \frac{d^{r-j}p_1}{dt^{r-j}}(t)\,(w_1-w_k)^j\right] e^{(w_1-w_k)t}$$

$= q_1(t)\, e^{(w_1-w_k)t}$, where $q_1(t)$ is the polynomial

$$\sum_{j=0}^{r} \binom{r}{j} \frac{d^{r-j}p_1}{dt^{r-j}}(t)\,(w_1-w_k)^j$$

Note $\deg\left(\frac{d^{r-j} p_1}{dt^{r-j}}\right) < \deg(p_1)$, unless $j = r$.

So if $p_1(t) = a_s t^s +$ lower degree terms,

$q_1(t) = a_s (w_1 - w_k)^r t^s +$ lower degree terms.

In particular, $\deg(q_1) = \deg(p_1)$.

So

$$0 = \frac{d^r}{dt^r}\left[p_1(t) e^{(w_1 - w_k)t} + \cdots + p_{k-1}(t) e^{(w_{k-1} - w_k)t} + p_k(t) \right]$$

$$= q_1(t) e^{(w_1 - w_k)t} + \cdots + q_{k-1}(t) e^{(w_{k-1} - w_k)t} + 0$$

where $\deg(q_k) = \deg(p_k)$.

By inductive hypothesis,

$q_1(t) = \cdots = q_{k-1}(t) = 0$, and $q_1, \dots, q_{k-1}$ are zero polynomials.

But $p_1, \dots, p_{k-1}$ have same degrees as $q_1, \dots, q_{k-1}$, respectively. So, $p_1, \dots, p_{k-1}$ are also zero polynomials.

Finally,

$$\underbrace{p_1(t) e^{w_1 t} + \cdots + p_{k-1}(t) e^{w_{k-1} t}}_{=0} + p_k(t) e^{w_k t} = 0$$

implies that $p_k(t) = 0$ for all $t \in I$

(after dividing by $e^{w_k t}$), so

Claim is true for k if it is true for $k-1$.

By induction, claim is true for all positive integers.

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