Mthe 237
Lecture 15
Oct. 13, 2017

Topic: Some qualitative properties of solutions of linear homogeneous equations with constant coefficients

---

Linear homogeneous equations

$$(H) \qquad \frac{d^r y}{dt^r} + a_{r-1}(t)\frac{d^{r-1}y}{dt^{r-1}} + \cdots + a_1(t)\frac{dy}{dt} + a_0(t)y = 0$$

$(a_k(t) \in \mathbb{R})$ continuous

can be split into two categories:

- Every $a_k$ is a constant ( $(H)$ is then said to have constant coefficients); or
- At least one of the $a_k$ is not a constant function of $t$
  ( $(H)$ is then said to have time-varying coefficients)

Constant:

$$\frac{d^2y}{dt^2} + 5y = 0$$

Time-varying:

$$\frac{d^2y}{dt^2} + ty = 0$$

(Airy Equation)

Whereas for constant coefficients we have a general procedure for writing down a basis of the space of solutions, for time-varying coefficients no such general method is known.

Frequently, solutions to equations with time-varying coefficients provably <u>cannot</u> be written in terms of familiar functions like exp, log, cos, sin, $\sqrt[n]{\ }$

and their compositions, sums, and products.

Instead, solutions are studied in more qualitative ways. The goal for today is to see some examples of qualitative properties.

---

**Theorem (Abel's Theorem)** Let $\varphi_1$ and $\varphi_2$ be two solutions of

$$\frac{d^2y}{dt^2} + a_1(t)\frac{dy}{dt} + a_0(t)\,y = 0 \quad \text{over } I.$$

Then, there exists a real constant $C$ so that for all $t \in I$,

$$W(\varphi_1, \varphi_2)(t) \overset{\text{def.}}{=} \det\begin{pmatrix} \varphi_1(t) & \varphi_2(t) \\ \frac{d\varphi_1}{dt}(t) & \frac{d\varphi_2}{dt}(t) \end{pmatrix}$$

$$= C\exp\left(-\int a_1(t)\,dt\right)$$

---

Proof: Write $\varphi'(t) = \frac{d\varphi}{dt}(t)$. By definition,

$$W(t) = \varphi_1(t)\,\varphi_2'(t) - \varphi_2(t)\,\varphi_1'(t).$$

Differentiating,

$$\frac{dW}{dt}(t) = \left(\varphi_1'(t)\,\varphi_2'(t) + \varphi_1(t)\,\varphi_2''(t)\right)$$
$$- \left(\varphi_2'(t)\,\varphi_1'(t) + \varphi_2(t)\,\varphi_1''(t)\right)$$

$$= \varphi_1(t)\varphi_2''(t) - \varphi_2(t)\varphi_1''(t).$$

Because $\varphi_1(t)$ is a solution of $y'' + a_1(t)y' + a_0(t)y = 0$,

$$\varphi_1''(t) = -a_1(t)\varphi_1'(t) - a_0(t)\varphi_1(t).$$

Similarly,

$$\varphi_2''(t) = -a_1(t)\varphi_2'(t) - a_0(t)\varphi_2(t).$$

Therefore,

$$\frac{dW}{dt}(t) = \varphi_1(t)\left(-a_1(t)\varphi_2'(t) - a_0(t)\varphi_2(t)\right)$$

$$- \varphi_2(t)\left(-a_1(t)\varphi_1'(t) - a_0(t)\varphi_1(t)\right)$$

$$= -a_1(t)\left[\varphi_1(t)\varphi_2'(t) - \varphi_2(t)\varphi_1'(t)\right]$$

$$- a_0(t)\left[\underbrace{\varphi_1(t)\varphi_2(t) - \varphi_2(t)\varphi_1(t)}_{=0}\right]$$

$$= -a_1(t)\, W(t).$$

Therefore, the Wronskian satisfies a separable differential equation. We can solve this as usual

$$\frac{1}{W}\frac{dW}{dt} = -a_1(t),$$

$$\int \frac{dW}{W} = \int -a_1(t)\,dt$$

$$\ln|W| + C = -\int a_1(t)\,dt$$

$$W(\varphi_1, \varphi_2)(t) = C\exp\left(-\int a_1(t)\,dt\right).$$

$\square$

**Example.** $\frac{d^2y}{dt^2} + y = 0.$ $\quad (a_1(t) = 0.)$

We know from before that $\sin(t), \cos(t)$ are solutions

Abel's Theorem says

$$W(\sin(t), \cos(t))(t) = C\exp(0) = C,$$

for some real constant $C$.

We computed before that in fact

$$W(\sin(t), \cos(t))(t) = -1.$$

Abel's Theorem generalizes to $r^{\text{th}}$ order equations as follows:

**Theorem (Abel's Theorem)** Let $\varphi_1, \varphi_2, \dots, \varphi_r$ be solutions of

$$\frac{d^ry}{dt^r} + a_{r-1}(t)\frac{d^{r-1}y}{dt^{r-1}} + \cdots + a_1(t)\frac{dy}{dt} + a_0(t)y = 0$$

over $I$.

Then there exists a real constant $C$ so that for all $t \in I$,

$$W(\varphi_1, \dots, \varphi_r)(t) = C \exp\left(-\int a_{r-1}(t)\,dt\right).$$

This version can again be proved with a computation, which we omit here.

Example: $\varphi_1 = e^t, \quad \varphi_2 = te^t, \quad \varphi_3 = t^2 e^t$

are solutions of a linear homogeneous differential equation with characteristic polynomial

$$\chi(z) = (z-1)^3$$

$$= z^3 - 3z^2 + 3z - 1$$

$$\frac{d^3y}{dt^3} - 3\frac{d^2y}{dt^2} + 3\frac{dy}{dt} - y = 0$$

$$a_{r-1}(t) = a_2(t) = -3.$$

Abel's Theorem says that

$$W(\varphi_1, \varphi_2, \varphi_3)(t) = C \exp\left(-\int (-3)\,dt\right)$$

$$= Ce^{3t}.$$

In fact, we computed on HW3 that $C=2$.

## § Change of Basis and Wronskians

Suppose that $g_1(t) = c_{11} f_1(t) + c_{12} f_2(t)$

$g_2(t) = c_{21} f_1(t) + c_{22} f_2(t)$

$c_{ij} \in \mathbb{R}$

$$\left( \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \right)$$

**Q**: How are $W(f_1, f_2)$ and $W(g_1, g_2)$ related?

$$\begin{pmatrix} g_1(t) & g_2(t) \\ g_1'(t) & g_2'(t) \end{pmatrix} = \begin{pmatrix} c_{11} f_1 + c_{12} f_2 & c_{21} f_1 + c_{22} f_2 \\ c_{11} f_1' + c_{12} f_2' & c_{21} f_1' + c_{22} f_2' \end{pmatrix}$$

$$= \begin{pmatrix} f_1 & f_2 \\ f_1' & f_2' \end{pmatrix} \begin{pmatrix} c_{11} & c_{21} \\ c_{12} & c_{22} \end{pmatrix}$$

$$= \begin{pmatrix} f_1 & f_2 \\ f_1' & f_2' \end{pmatrix} \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}^t$$

"t" denotes matrix <u>transpose</u>

Therefore,

$$W(g_1, g_2)(t) = \det \begin{pmatrix} g_1 & g_2 \\ g_1' & g_2' \end{pmatrix}$$

$$= \det \left[ \begin{pmatrix} f_1 & f_2 \\ f_1' & f_2' \end{pmatrix} \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}^t \right]$$

$$= \det \begin{pmatrix} f_1 & f_2 \\ f_1' & f_2' \end{pmatrix} \det \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}$$

$$= W(f_1, f_2)(t) \cdot \left( \begin{matrix} \text{Real} \\ \text{Constant} \end{matrix} \right)$$

For $r$ functions, one can check that similarly

If $\begin{pmatrix} g_1 \\ \vdots \\ g_r \end{pmatrix} = C \begin{pmatrix} f_1 \\ \vdots \\ f_r \end{pmatrix}$, $C$ $r \times r$ real matrix, then

$$W(g_1, \dots, g_r)(t) = W(f_1, \dots, f_r)(t) \cdot \det(C).$$

In Abel's Theorem, the expression

$\exp(-\int a_{r-1}(t)\,dt)$ depends on the differential equation only, and not the choice of the set $\{\varphi_1, \dots, \varphi_r\}$ of solutions. The appearance of the constant $C$ in

$$W(\varphi_1, \dots, \varphi_r)(t) = C \exp\left(-\int a_{r-1}(t)\,dt\right)$$

reflects the fact that changing the choice of $\{\varphi_1, \dots, \varphi_r\}$ as above scales the Wronskian by a constant.

§

**Prop.** Let $\varphi_1, \dots, \varphi_r$ be solutions of

$$\frac{d^r y}{dt^r} + a_{r-1}(t) \frac{d^{r-1} y}{dt^{r-1}} + \dots + a_1(t) \frac{dy}{dt} + a_0(t) y = 0$$

over $I$.

Then either:

i) $W(\varphi_1, \dots, \varphi_r)(t) = 0$ for all $t \in I$, or

ii) $W(\varphi_1, \dots, \varphi_r)(t) \neq 0$ for all $t \in I$.

**Proof:** By Abel's formula,

$$W(\varphi_1, \dots, \varphi_r)(t) = C\exp\left(-\int a_{r-1}(t)\, dt\right)$$

for some $C \in \mathbb{R}$.

If there exists $t_0 \in I$ with $W(t_0) = 0$, then, since $\exp(\text{any function of } t) > 0$ for all $t$, we must have $C = 0$.

Thus, $W(\varphi_1, \dots, \varphi_r)(t) = 0 \cdot \exp\left(-\int a_{r-1}(t) dt\right)$
$= 0$ for all $t$.

Otherwise, there does not exist a $t_0 \in I$ with $W(t_0) = 0$, so that

$$W(\varphi_1, \dots, \varphi_r)(t) \neq 0 \quad \text{for all } t.$$

$\square$

For the next theorem, we need to remind ourselves of the following theorem:

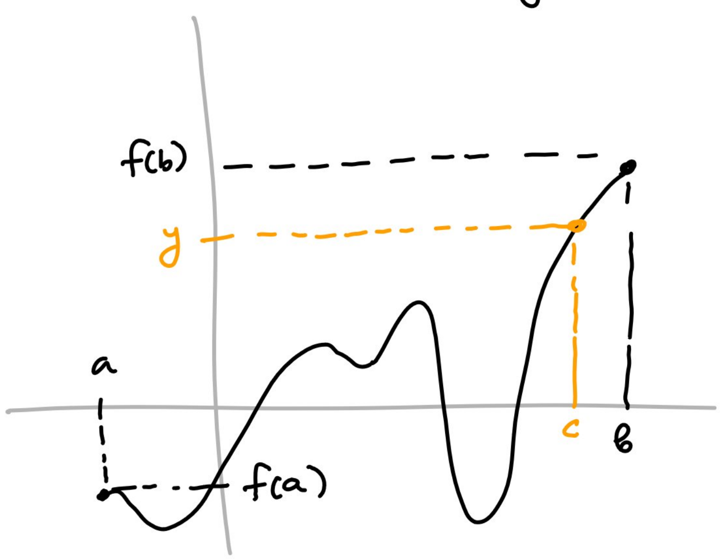
**Theorem [Intermediate Value Theorem]**

Let $f$ be a continuous function $f : [a,b] \to \mathbb{R}$.

Then for every real number $y$ between $f(a)$ and $f(b)$ (meaning
$$\min(f(a), f(b)) \le y \le \max(f(a), f(b)))$$

There exists a $c \in [a,b]$ with

$$f(c) = y.$$




Intuitively,

" If one moves continuously between two points on a line, one passes through every point between them. "

Theorem [Sturm Separation Theorem]

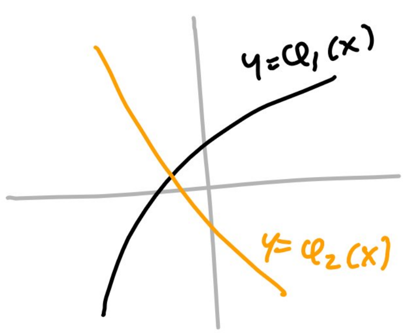
Let $\varphi_1(t)$ and $\varphi_2(t)$ be two linearly independent solutions of

$$(*) \qquad \frac{d^2y}{dt^2} + a_1(t)\frac{dy}{dt} + a_0(t)y = 0 \qquad \text{over } I.$$

Between any two successive zeros of one of the solutions there exists a zero of the other solution.

We can read the theorem as saying that the zeros of $\varphi_1$ and $\varphi_2$ alternate.
This restricts the pairs of functions that can occur as solutions of $(*)$.

Some possible graphs of pairs of solutions:





No oscillation
– Theorem doesn't say anything about this case

Oscillations over one interval, no oscillations over another interval

One solution can't oscillate without the other solution oscillating also.

Proof: Because $\varphi_1$ and $\varphi_2$ are linearly independent, there exists a $t_0 \in I$ such that
$$W(\varphi_1, \varphi_2)(t_0) \neq 0.$$

Therefore, $W(\varphi_1, \varphi_2)(t) \neq 0$ for all $t \in I$ (by earlier proposition from today).

Either $W(t) > 0$ for all $t$ or $W(t) < 0$ for all $t$.
Suppose $> 0$ (other case is similar).

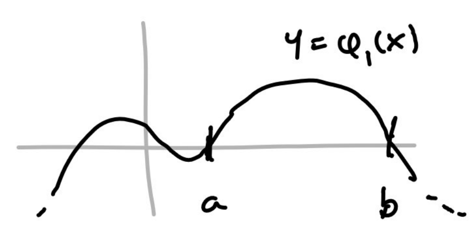
Let $a, b \in I$ be successive zeros of $\varphi_1$, so that
$$\varphi_1(a) = \varphi_1(b) = 0, \quad \text{and}$$
$$\varphi_1(t) \neq 0 \quad \text{for all } t \in (a, b).$$

$$W(\varphi_1, \varphi_2)(a) = \varphi_1(a)\varphi_2'(a) - \varphi_2(a)\varphi_1'(a)$$
$$= -\varphi_2(a)\varphi_1'(a) > 0$$

$$W(\varphi_1, \varphi_2)(b) = \varphi_1(b)\varphi_2'(b) - \varphi_2(b)\varphi_1'(b)$$
$$= -\varphi_2(b)\varphi_1'(b) > 0$$

Because $a, b$ are successive zeros of $\varphi_1$, the derivative of $\varphi_1$ must change sign:

$$\varphi_1'(a) = -\varphi_1'(b)$$



Note if $\varphi_1'(a) = 0$, then since

$\varphi_1(a) = \varphi_1'(a) = 0$, by Existence and Uniqueness Theorem $\varphi_1(t) = 0$ for all $t$, which contradicts $\varphi_1$ and $\varphi_2$ being linearly independent.

Alternatively, $\varphi_1'(a) \neq 0$ because $W(a) = -\varphi_1'(a)\varphi_2(a) > 0$

Similar remarks apply to $\varphi_1'(b)$.

Since $\varphi_1'(a)$ and $\varphi_1'(b)$ have opposite sign, and

$-\varphi_1'(a)\varphi_2(a) > 0$ and $-\varphi_1'(b)\varphi_2(b) > 0$ have same sign,

$\varphi_2(a)$ and $\varphi_2(b)$ have opposite sign.

Therefore, $\varphi_2(c) = 0$ for some $c \in (a,b)$ by the Intermediate Value Theorem.

This shows the theorem for $\varphi_1$. Now interchange $\varphi_1$ and $\varphi_2$ in the above argument to obtain same conclusion for $\varphi_2$. ∎