Mthe 237
Lecture 14
Oct. 11, 2017

Topic: Polynomial Differential Operators with Constant Coefficients

PRELIMINARIES

1. Functions $I \subset \mathbb{R} \longrightarrow \mathbb{C}$

A function $f : I \longrightarrow \mathbb{C}$ is given by a pair of functions $x, y : I \longrightarrow \mathbb{R}$ (x and y are real-valued functions on I). By definition,

$$f(t) = x(t) + i\,y(t), \qquad t \in I.$$
$$\in \mathbb{C}$$

We can think of this as a parametrized path in the complex plane $\mathbb{C}$.

If x and y are differentiable functions, then we define

$$\frac{df}{dt}(t) = \frac{dx}{dt}(t) + i\,\frac{dy}{dt}, \qquad t \in I.$$

Thinking of f as a path, $\frac{df}{dt}(t)$ is the velocity vector at time t.

Aside:

The derivative of complex-valued functions inherits the usual differentiation rules from x and y. For instance, if $f(t) = x(t) + i\,y(t)$ and $g(t) = u(t) + i\,v(t)$,

then $(fg)(t) = \big(x(t)u(t) - y(t)v(t)\big) + i\,\big(x(t)v(t) + y(t)u(t)\big)$

then $\dfrac{d(fg)}{dt}(t)$

$$= \begin{pmatrix} x'(t)u(t) + x(t)u'(t) \\ - y'(t)v(t) - y(t)v'(t) \end{pmatrix} + i \begin{pmatrix} x'(t)v(t) + x(t)v'(t) \\ + y'(t)u(t) + y(t)u'(t) \end{pmatrix}$$

$$= x'(t)\left(u(t) + iv(t)\right) + x(t)\left(u'(t) + iv'(t)\right) + iy'(t)\left(u(t) + iv(t)\right) + iy(t)\left(u'(t) + iv'(t)\right)$$

$$= \left(x'(t) + iy'(t)\right)\left(u(t) + iv(t)\right) + \left(x + iy(t)\right)\left(u'(t) + iv'(t)\right)$$

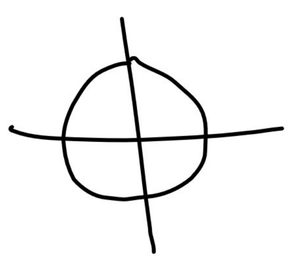
$$= \frac{df}{dt}(t)\, g(t) + f(t)\frac{dg}{dt}(t).$$

Example:

The image of the function

$$f : t \longmapsto \underbrace{\cos(t)}_{x(t)} + i\underbrace{\sin(t)}_{y(t)} = e^{it}, \quad t \in [0, 2\pi)$$

is the unit circle centered at $(0,0)$ in $\mathbb{C}$.



The derivative is

$$\frac{df}{dt}(t) = -\sin(t) + i\cos(t) = i\left(\cos(t) + i\sin(t)\right)$$
$$= ie^{it}.$$

Since multiplication by $i = e^{i\frac{\pi}{2}}$ rotates vectors in $\mathbb{C}$ by $\frac{\pi}{2}$, we see that for this path the velocity ie^{it} is always perpendicular to the position e^{it}.

Higher-order derivatives are defined by iterating the first-order derivative. We have

$$\frac{d^r f}{dt^r}(t) = \frac{d^r x}{dt^r} + i\,\frac{d^r y}{dt^r},$$

assuming $x, y \in C^r(I, \mathbb{R})$.

We can now make sense of the statement that $\varphi : I \to \mathbb{C}$ is a solution of

$$\frac{d^r y}{dt^r} + a_{r-1}(t)\frac{d^{r-1} y}{dt^{r-1}} + \cdots + a_1(t)\frac{dy}{dt} + a_0(t)\,y = 0,$$

where $a_k(t)$ are continuous functions $I \to \mathbb{C}$

($f(t) = x(t) + iy(t)$ is continuous if and only if x, y are continuous)

Def.

$$C^r(I, \mathbb{C}) = \left\{ f : I \to \mathbb{C} \;\middle|\; f, \frac{df}{dt}, \ldots, \frac{d^r f}{dt^r} \text{ exist and are continuous} \right\}$$

$$C^\infty(I, \mathbb{C}) = \bigcap_{r=0}^{\infty} C^r(I, \mathbb{C}) = \left\{ f : I \to \mathbb{C} \;\middle|\; f, \frac{df}{dt}, \ldots \text{ exist and are continuous} \right\}$$

2. Polynomial Differential Operators with Constant Coefficients

Def. A polynomial differential operator (with constant coefficients) is a formal expression of the form

$$(*) \qquad \frac{d^r}{dt^r} + a_{r-1}\frac{d^{r-1}}{dt^{r-1}} + \cdots + a_1\frac{d}{dt} + a_0, \qquad a_k \in \mathbb{C}.$$

For short, we often write

$$p\left(\frac{d}{dt}\right)$$

instead of $(*)$, where p is the polynomial

$$p(z) = z^r + a_{r-1}z^{r-1} + \cdots + a_1 z + a_0$$

(This notation will be justified shortly, once we define the sum and product of polynomial differential operators.)

We think of $p\left(\frac{d}{dt}\right)$ as acting on functions, defining a linear map

$$C^{k+r}(I, \mathbb{C}) \longrightarrow C^k(I, \mathbb{C})$$

$$y(t) \longmapsto \frac{d^r y}{dt^r} + a_{r-1}\frac{d^{r-1}y}{dt^{r-1}} + \cdots + a_0 y.$$

Examples:

- $\left(\frac{d^2}{dt^2}+1\right)y = \frac{d^2y}{dt^2}+y$
- $\left(\frac{d^3}{dt^3}+3\frac{d}{dt}\right)y = \frac{d^3y}{dt^3}+3\frac{dy}{dt}$

The fact that $p\left(\frac{d}{dt}\right)$ defines a linear map

$$C^{k+r}(I,\mathbb{C}) \longrightarrow C^k(I,\mathbb{C})$$

explicitly means that

$$\boxed{\begin{aligned} &p\left(\tfrac{d}{dt}\right)\left(c_1y_1+c_2y_2\right) = c_1\,p\left(\tfrac{d}{dt}\right)(y_1) + c_2\,p\left(\tfrac{d}{dt}\right)(y_2), \\ &\qquad\text{for any } c_1,c_2\in\mathbb{C} \\ &\qquad\qquad\quad y_1,y_2\in C^{k+r}(I,\mathbb{C}). \end{aligned}}$$

This property follows from the linearity of

$$\frac{d}{dt},\ \dots,\ \frac{d^r}{dt^r}.$$

Sum of Polynomial Differential Operators:

Def. For any y (appropriately differentiable),

$$\left[\left(\frac{d^r}{dt^r}+a_{r-1}\frac{d^{r-1}}{dt^{r-1}}+\dots+a_1\frac{d}{dt}+a_0\right) + \left(\frac{d^s}{dt^s}+b_{s-1}\frac{d^{s-1}}{dt^{s-1}}+\dots+b_1\frac{d}{dt}+b_0\right)\right]y = \dots$$

$$\dots = \frac{d^ry}{dt^r}+\dots+a_0y+\frac{d^sy}{dt^s}+\dots+b_0y.$$

<u>Reminder</u>: For polynomials $p(z) = z^r + \cdots + a_1 z + a_0$
$q(z) = z^s + \cdots + b_1 z + b_0$,

$$(p+q)(z) = p(z) + q(z)$$
$$= z^r + \cdots + a_1 z + a_0 + z^s + \cdots + b_1 z + b_0.$$

<u>Prop</u>. For any pair of polynomial differential operators $p(\frac{d}{dt})$, $q(\frac{d}{dt})$,

$$\left[p(\tfrac{d}{dt}) + q(\tfrac{d}{dt}) \right] y = (p+q)(\tfrac{d}{dt})\, y$$ for any y (sufficiently differentiable)

This proposition is clear from the definitions:

If $p(z) = z^r + \cdots + a_1 z + a_0$
$q(z) = z^s + \cdots + b_1 z + b_0$,

$$\left[p(\tfrac{d}{dt}) + q(\tfrac{d}{dt}) \right] y$$
$$= \left[\frac{d^r}{dt^r} + \cdots + a_0 + \frac{d^s}{dt^s} + \cdots + b_0 \right] y$$
$$= \frac{d^r y}{dt^r} + \cdots + a_0 y + \frac{d^s y}{dt^s} + \cdots + b_0 y,$$ whereas

$$(p+q)(z) = z^r + \cdots + a_0 + z^s + \cdots + b_0,$$ so

$$(p+q)(\tfrac{d}{dt})\, y = \frac{d^r y}{dt^r} + \cdots + a_0 y + \frac{d^s y}{dt^s} + \cdots + b_0 y.$$

Product of Polynomial Differential Operators:

Def. For any y (sufficiently differentiable),

$$\left[\left(\frac{d^r}{dt^r} + a_{r-1}\frac{d^{r-1}}{dt^{r-1}} + \cdots + a_1\frac{d}{dt} + a_0\right)\left(\frac{d^s}{dt^s} + b_{s-1}\frac{d^{s-1}}{dt^{s-1}} + \cdots + b_0\right)\right]y$$

$$\overset{\text{def.}}{=} \left(\frac{d^r}{dt^r} + \cdots + a_1\frac{d}{dt} + a_0\right)\left[\left(\frac{d^s}{dt^s} + \cdots + b_1\frac{d}{dt} + b_0\right)y\right]$$

In words, product of two differential operators is given by composition.

Reminder: For polynomials

$$p(z) = z^r + \cdots + a_1 z + a_0$$
$$q(z) = z^s + \cdots + b_1 z + b_0,$$

$$(pq)(z) = p(z)q(z)$$

Example:

$$p(z) = z^2+1$$
$$q(z) = 3z+4$$

$$(pq)(z) = p(z)q(z) = (z^2+1)(3z+4)$$
$$= 3z^3 + 4z^2 + 3z + 4$$

The following is not crucial:

In general, writing $p(z) = a_r z^r + \cdots + a_1 z + a_0$ with $a_r = 1$

$q(z) = b_s z^s + \cdots + b_1 z + b_0$ with $b_s = 1$,

$$(pq)(z) = (a_r b_s) z^{r+s} + (a_r b_{s-1} + a_{r-1} b_s) z^{r+s-1} + \cdots$$

$$\cdots + (a_r b_{s-2} + a_{r-1} b_{s-1} + a_{r-2} b_s) z^{r+s-2} + \cdots$$

$$\cdots + (a_1 b_0 + a_0 b_1) z + (a_0 b_0)$$

$$= \sum_{d=0}^{r+s} \left(\sum_{k=0}^{d} a_{d-k} b_k \right) z^d$$

<u>Prop.</u> For any pair of polynomial differential operators with <u>constant</u> coefficients $p(\frac{d}{dt})$, $q(\frac{d}{dt})$,

$$p\left(\frac{d}{dt}\right)\left[q\left(\frac{d}{dt}\right) y\right] = (pq)\left(\frac{d}{dt}\right) y$$

Let's check this for an example:

$$p(z) = z^2 + 1, \qquad q(z) = 3z + 4$$

$$(pq)(z) = 3z^3 + 4z^2 + 3z + 4 \quad \text{(computed above)}$$

We have

$$p\left(\frac{d}{dt}\right)\left[q\left(\frac{d}{dt}\right) y\right] = \left(\frac{d^2}{dt^2} + 1\right)\left[\left(3\frac{d}{dt} + 4\right) y\right]$$

$$= \left(\frac{d^2}{dt^2} + 1\right)\left[3\frac{dy}{dt} + 4y\right]$$

$$= \frac{d^2}{dt^2}\left(3\frac{dy}{dt} + 4y\right) + 1\cdot\left(3\frac{dy}{dt} + 4y\right)$$

$$= \frac{d^2}{dt^2}\left(3\frac{dy}{dt}\right) + \frac{d^2}{dt^2}(4y) + \left(3\frac{dy}{dt} + 4y\right)$$

$$= 3\frac{d^3y}{dt^3} + 4\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 4y$$

Crucial to have constant coefficients here.

$$= \left(3\frac{d^3}{dt^3} + 4\frac{d^2}{dt^2} + 3\frac{d}{dt} + 4\right)y$$

$$= (pq)\left(\frac{d}{dt}\right)y$$

General computation (not important):

$p(z) = a_r z^r + \cdots + a_0$, with $a_r = 1$

$q(z) = b_s z^s + \cdots + b_0$, with $b_s = 1$.

$$p\left(\frac{d}{dt}\right)\left[q\left(\frac{d}{dt}\right)y\right] = \left(a_r\frac{d^r}{dt^r} + \cdots + a_0\right)\left[b_s\frac{d^sy}{dt^s} + \cdots + b_0y\right]$$

$$= a_r\frac{d^r}{dt^r}\left(b_s\frac{d^sy}{dt^s} + \cdots + b_0y\right) + \cdots + a_0\left(b_s\frac{d^sy}{dt^s} + \cdots + b_0y\right)$$

$$= (a_rb_s)\frac{d^{r+s}y}{dt^{r+s}} + (a_rb_{s-1} + a_{r-1}b_s)\frac{d^{r+s-1}y}{dt^{r+s-1}} + \cdots + a_0b_0y$$

$$= (pq)\left(\frac{d}{dt}\right)y.$$

Corollary For any pair $p(\frac{d}{dt})$, $q(\frac{d}{dt})$ of polynomial differential operators with constant coefficients and any y (sufficiently differentiable),

$$p\left(\frac{d}{dt}\right)\left[q\left(\frac{d}{dt}\right)y\right] = q\left(\frac{d}{dt}\right)\left[p\left(\frac{d}{dt}\right)y\right]$$

Proof: By the previous proposition,

$$p\left(\frac{d}{dt}\right)\left[q\left(\frac{d}{dt}\right)y\right] = (pq)\left(\frac{d}{dt}\right)y \quad \text{and}$$

$$q\left(\frac{d}{dt}\right)\left[p\left(\frac{d}{dt}\right)y\right] = (qp)\left(\frac{d}{dt}\right)y.$$

Because $(pq) = (qp)$ as polynomials (this can be checked using the general expression for the product of two polynomials above, for instance), the claim follows. ∎

The corollary says that $p(\frac{d}{dt})$ and $q(\frac{d}{dt})$ may be applied in either order. We say $p(\frac{d}{dt})$ and $q(\frac{d}{dt})$ commute.

Remark. The corollary is false if we allow time-varying coefficients. For instance,

$$t\frac{d}{dt}\left[\frac{d}{dt}y\right] = t\frac{d}{dt}\left[\frac{dy}{dt}\right] = t\frac{d^2y}{dt^2}, \quad \text{whereas}$$

$$\frac{d}{dt}\left[t\frac{d}{dt}y\right] = \frac{d}{dt}\left[t\frac{dy}{dt}\right] \underset{\text{Product rule.}}{=} \frac{dy}{dt} + t\frac{d^2y}{dt^2}$$

APPLICATION TO LINEAR HOMOGENEOUS DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

The equation

$$\frac{d^r y}{dt^r} + a_{r-1}\frac{d^{r-1}y}{dt^{r-1}} + \cdots + a_1 \frac{dy}{dt} + a_0 y = 0, \quad a_k \in \mathbb{R}$$

has characteristic polynomial

$$\chi(z) = z^r + a_{r-1} z^{r-1} + \cdots + a_1 z + a_0$$

and may be written as

$$\chi\left(\frac{d}{dt}\right) y = 0, \qquad \text{for short.}$$

If $\chi(z)$ factors as

$$\chi(z) = (z-r_1)^{m_1} \cdots (z-r_k)^{m_k}\left[(z-z_1)(z-\bar{z}_1)\right]^{n_1} \cdots \left[(z-z_\ell)(z-\bar{z}_\ell)\right]^{n_\ell}$$

Then the operator $\chi\left(\frac{d}{dt}\right)$ factors as

$$\chi\left(\frac{d}{dt}\right) = \left(\frac{d}{dt}-r_1\right)^{m_1} \cdots \left(\frac{d}{dt}-r_k\right)^{m_k}\left[\left(\frac{d}{dt}-z_1\right)\left(\frac{d}{dt}-\bar{z}_1\right)\right]^{n_1} \cdots \left[\left(\frac{d}{dt}-z_\ell\right)\left(\frac{d}{dt}-\bar{z}_\ell\right)\right]^{n_\ell}$$

This follows from the compatibility of the sum and product of polynomial differential operators with constant coefficients with sum and product of polynomials (the two propositions above).

By linearity of polynomial differential operators,

$$p\left(\frac{d}{dt}\right) \underset{\substack{\uparrow \\ \text{zero function.}}}{0} = 0 \qquad \text{for any } p.$$

Therefore, to check that $\varphi : I \longrightarrow \mathbb{C}$ is a solution of

$$\chi\left(\frac{d}{dt}\right) y = 0,$$

it is enough to check that

$$\left(\frac{d}{dt} - r_j\right)^{m_j} y = 0 \qquad \text{for one of the real roots } r_j \text{ of } \chi(z)$$

or

$$\left(\frac{d}{dt} - z_j\right)^{n_j} y = 0$$

or

$$\left(\frac{d}{dt} - \overline{z_j}\right)^{n_j} y = 0$$

for one of the complex roots z_j of $\chi(z)$.

The following lemma will be useful:

<u>lemma</u>. Let $w \in \mathbb{C}$ and $f \in C^m(I, \mathbb{C})$ $(m = 1, 2, \ldots)$

Then

$$\left(\frac{d}{dt} - w\right)^m f(t) e^{wt} = \frac{d^m f}{dt^m}(t)\, e^{wt}$$

Proof of Lemma

By induction on m.

For $m=1$:

$$\left(\frac{d}{dt}-w\right) f(t)e^{wt}$$

$$= \frac{d}{dt}\left(f(t)e^{wt}\right) - wf(t)e^{wt}$$

$$= \frac{df}{dt}(t)e^{wt} + wf(t)e^{wt} - wf(t)e^{wt}$$

$$= \frac{df}{dt}(t)e^{wt},$$ verifying the claim for $m=1$.

Now suppose true for $m-1$. We show true for m.

$$\left(\frac{d}{dt}-w\right)^m f(t)e^{wt}$$

$$= \left(\frac{d}{dt}-w\right)\left[\left(\frac{d}{dt}-w\right)^{m-1} f(t)e^{wt}\right]$$

$$= \left(\frac{d}{dt}-w\right)\left(\frac{d^{m-1}f}{dt^{m-1}}(t)e^{wt}\right)$$

$$= \frac{d}{dt}\left(\frac{d^{m-1}f}{dt^{m-1}}(t)e^{wt}\right) - w\frac{d^{m-1}f}{dt^{m-1}}(t)e^{wt}$$

$$= \frac{d^mf}{dt^m}(t)e^{wt} + w\frac{d^{m-1}f}{dt^{m-1}}(t)e^{wt} - w\frac{d^{m-1}f}{dt^{m-1}}(t)e^{wt}$$

$$= \frac{d^mf}{dt^m}(t)e^{wt},$$ verifying the claim for m assuming true for $m-1$.

The claim follows by induction. $\square$

We are now ready to check that the functions produced by our procedure for constructing a basis of solutions are indeed solutions.

For real root r_j of $\chi(z)$:

Apply lemma to $f(t) = t^k e^{r_j t}$, $k = 0, 1, \ldots, m_j - 1$.

$$\left(\frac{d}{dt} - r_j\right)^{m_j} \left(t^k e^{r_j t}\right)$$

$$= \frac{d^{m_j}(t^k)}{dt^{m_j}} e^{r_j t} = 0 \cdot e^{r_j t} = 0.$$

Therefore, $\chi\left(\frac{d}{dt}\right)\left(t^k e^{r_j t}\right) = 0$.

$$e^{r_j t},\ t e^{r_j t},\ t^2 e^{r_j t},\ \ldots,\ t^{m_j - 1} e^{r_j t}$$

are solutions of $\chi\left(\frac{d}{dt}\right) y = 0$.

For complex roots z_j and $\overline{z_j}$ of $\chi(z)$:

Apply lemma to $f(t) = t^k e^{z_j t}$, $k = 0, 1, \ldots, n_j - 1$

$$\left(\frac{d}{dt} - z_j\right)^{n_j} \left(t^k e^{z_j t}\right)$$

$$= \frac{d^{n_j}(t^k)}{dt^{n_j}} e^{z_j t} = 0 \cdot e^{z_j t} = 0.$$

Similarly, $\left(\frac{d}{dt}-\overline{z_j}\right)^{n_j}\left(t^k e^{\overline{z_j}t}\right)=0.$

Therefore, $\chi\left(\frac{d}{dt}\right)\left(t^k e^{z_j t}\right)=0$ and

$$\chi\left(\frac{d}{dt}\right)\left(t^k e^{\overline{z_j}t}\right)=0$$

$$e^{z_j t},\ t e^{z_j t}, \dots,\ t^{n_j-1}e^{z_j t},$$

$$e^{\overline{z_j}t},\ t e^{\overline{z_j}t}, \dots,\ t^{n_j-1}e^{\overline{z_j}t}$$ are solutions of $\chi\left(\frac{d}{dt}\right)y=0.$

Similarly to the real-valued case, the complex-valued solutions $\varphi: I \to \mathbb{C}$ of $\chi\left(\frac{d}{dt}\right)y=0$ form a complex vector space.

Therefore, writing $z_j=\sigma_j+i\omega_j$,

$$t^k\left(\frac{e^{z_j t}+e^{\overline{z_j}t}}{2}\right)$$

$$=t^k\left(\frac{e^{\sigma_j t}e^{i\omega_j t}+e^{\sigma_j t}e^{-i\omega_j t}}{2}\right)$$

$$=t^k e^{\sigma_j t}\left(\frac{e^{i\omega_j t}+e^{-i\omega_j t}}{2}\right)=t^k e^{\sigma_j t}\cos(\omega_j t)$$

are again solutions of $\chi\left(\frac{d}{dt}\right)y=0$ (being a linear combination of solutions) and, similarly,

$$t^k \left(\frac{e^{z_j t} - e^{\overline{z_j} t}}{2i} \right)$$

$$= t^k \left(\frac{e^{\sigma_j t} e^{i\omega_j t} - e^{\sigma_j t} e^{-i\omega_j t}}{2i} \right)$$

$$= t^k e^{\sigma_j t} \left(\frac{e^{i\omega_j t} - e^{-i\omega_j t}}{2i} \right) = t^k e^{\sigma_j t} \sin(\omega_j t)$$

are again solutions of $\chi(\frac{d}{dt}) y = 0$.

This verifies that for each complex root $z_j = \sigma_j + i\omega_j$ of $\chi(z)$, the functions

$$e^{\sigma_j t} \cos(\omega_j t),\ t e^{\sigma_j t} \cos(\omega_j t), \dots,\ t^{n_j - 1} e^{\sigma_j t} \cos(\omega_j t),$$
$$e^{\sigma_j t} \sin(\omega_j t),\ t e^{\sigma_j t} \sin(\omega_j t), \dots,\ t^{n_j - 1} e^{\sigma_j t} \sin(\omega_j t)$$

are solutions of the differential equation

$$\chi(\tfrac{d}{dt}) y = 0.$$