Mthe 237
Lecture 13
Oct. 10, 2017

Topic: A closer look at solutions to first- and second-order linear homogeneous equations with constant coefficients

First Order

A general linear homogeneous equation with constant coefficients of first order has the form

$$\frac{dy}{dt} - a_0 y = 0 \quad , \quad a_0 \in \mathbb{R}$$

(This is not quite consistent with $\frac{dy}{dt} + a_0 y = 0$, (the difference is the sign of a_0))

Its characteristic polynomial is

$$\chi(z) = z - a_0.$$

This has a single real root, namely $z = a_0$.
Therefore (by the general procedure),

$$\{e^{a_0 t}\} \text{ is a basis of solutions.}$$

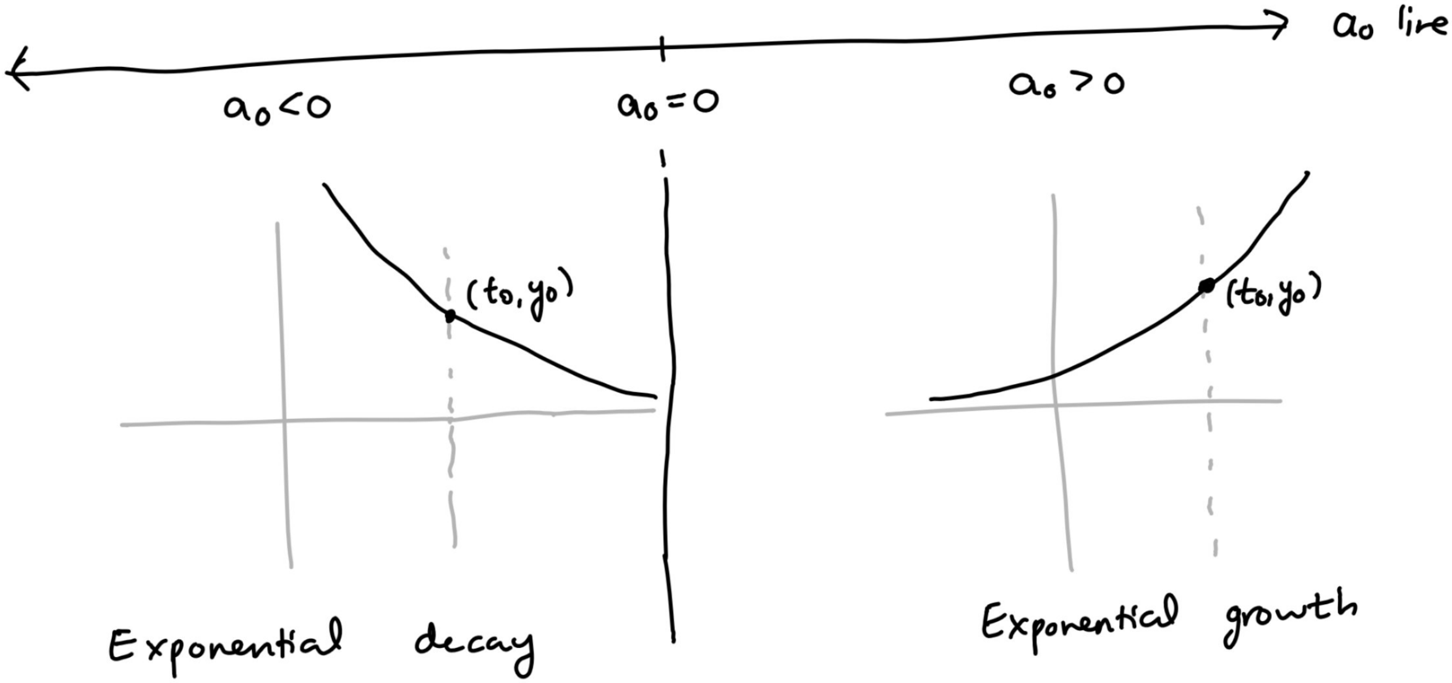
Every solution has the form $y(t) = Ce^{a_0 t}$, $C \in \mathbb{R}$.

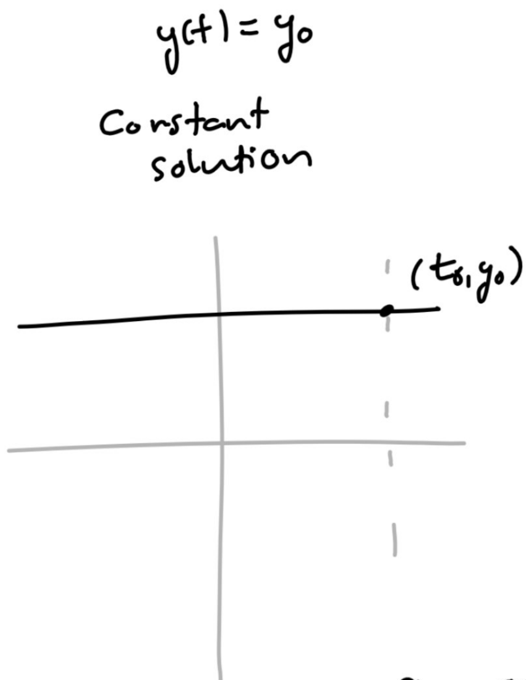
Imposing the initial condition $y(t_0) = y_0$, C is determined:

$$y_0 = y(t_0) = Ce^{a_0 t_0}, \quad \text{so} \quad C = y_0 e^{-a_0 t_0}$$

$$y(t) = y_0 e^{a_0(t - t_0)}.$$

The qualitative behaviour of the solutions depends on the sign of the coefficient a_0:





a_0 can be thought of as the rate of growth/decay

Example of Exponential Decay:

On average, a radioactive substance undergoes decay at a speed proportional to the amount of substance that remains. This process is modeled fairly well by the equation

$$\frac{dy}{dt} = a_0 y, \qquad a_0 < 0, \qquad y(0) = y_0.$$

(The solution is $y(t) = y_0 e^{a_0 t}$.)

Suppose that the substance is known to have half-life h (this is the time for the amount of substance to decay to half the initial amount) we can then find a_0:

$$\frac{y_0}{2} = y(h) = y_0 e^{a_0 h}$$

Thus, $\frac{1}{2} = e^{a_0 h}$, so $a_0 = \frac{\ln(\frac{1}{2})}{h}$

(Notice this is indeed negative, since $\ln(\frac{1}{2})$ is)

$$\begin{aligned} y(t) &= y_0 e^{\ln(\frac{1}{2})\frac{t}{h}} \\ &= y_0 \left(e^{\ln(\frac{1}{2})}\right)^{t/h} \\ &= y_0 \left(\frac{1}{2}\right)^{t/h} \end{aligned}$$

Problem: Uranium - 235 has a half-life of $7 \cdot 10^8$ years. How long for the sample to decay to a third of its initial size?

$$\frac{y_0}{3} = y(t) = y_0 \, e^{\ln(\frac{1}{2}) t/h}$$

$$\ln\left(\tfrac{1}{3}\right) = \ln\left(\tfrac{1}{2}\right)\frac{t}{h}$$

$$t = h \cdot \frac{\ln(\frac{1}{3})}{\ln(\frac{1}{2})} = (7 \cdot 10^8)\frac{\ln(\frac{1}{3})}{\ln(\frac{1}{2})} \approx 1.1 \cdot 10^9 \text{ years.}$$

Example of Exponential Growth:

Funds that are accruing continuously compounded interest at a rate of k per annum satisfy the differential equation

$$\frac{dy}{dt} = a_0 y, \quad a_0 > 0.$$

Example Suppose that an initial balance of 100 dollars accrues continuously compounded interest at a rate of 4% per annum. What will the balance be after five years?

$$y(t) = 100\, e^{0.04t}$$

$$y(5) = 100 e^{0.2} \approx 122 \text{ dollars.}$$

Second Order:

A general second-order linear homogeneous differential equation with constant coefficients may be written as

$$\frac{d^2y}{dt^2} + 2a_1 \frac{dy}{dt} + a_0 y = 0, \qquad a_1, a_0 \in \mathbb{R}.$$

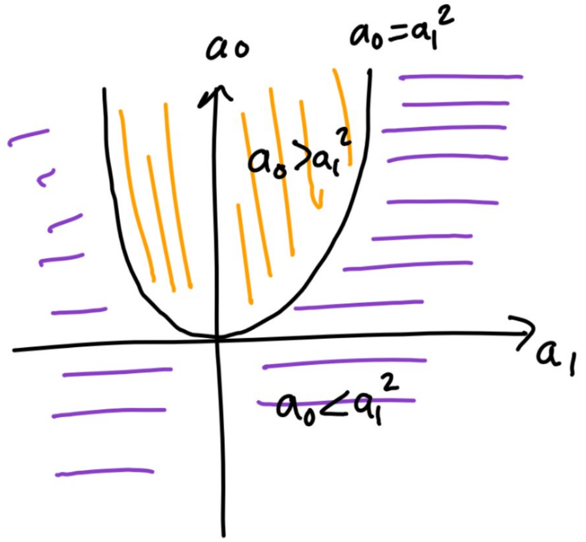
Its characteristic polynomial is

$$\chi(z) = z^2 + 2a_1 z + a_0.$$

The roots of the characteristic polynomial may be found using the quadratic formula

$$\frac{-2a_1 \pm \sqrt{4a_1^2 - 4 \cdot 1 \cdot a_0}}{2} = -a_1 \pm \sqrt{a_1^2 - a_0}.$$

The qualitative behaviour of the solutions depends on the sign of the discriminant $4a_1^2 - 4a_0$ (which has the same sign as $a_1^2 - a_0$.)



$a_1^2 - a_0 = 0$ describes a parabola in the (a_1, a_0)-plane.

The qualitative behaviour of the solutions depends on which region,

- The orange region above the parabola
- The parabola itself, or
- The purple region under the parabola,

the point (a_1, a_0) lies in.

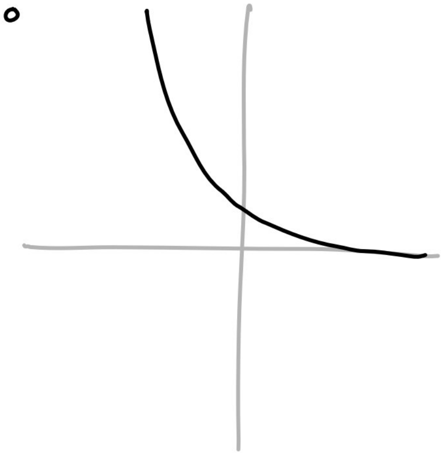
- If $a_1^2 - a_0 > 0$ $\quad (\Leftrightarrow a_0 < a_1^2)$,

  $\chi(z)$ has two distinct real roots

  $$r_1 = -a_1 + \sqrt{a_1^2 - a_0} \text{ and}$$
  $$r_2 = -a_1 - \sqrt{a_1^2 - a_0}$$

A basis of the space of solutions is $\left\{ e^{(-a_1 + \sqrt{a_1^2 - a_0})t}, e^{(-a_1 - \sqrt{a_1^2 - a_0})t} \right\}$

The graph of a typical solution loo like an exponential.

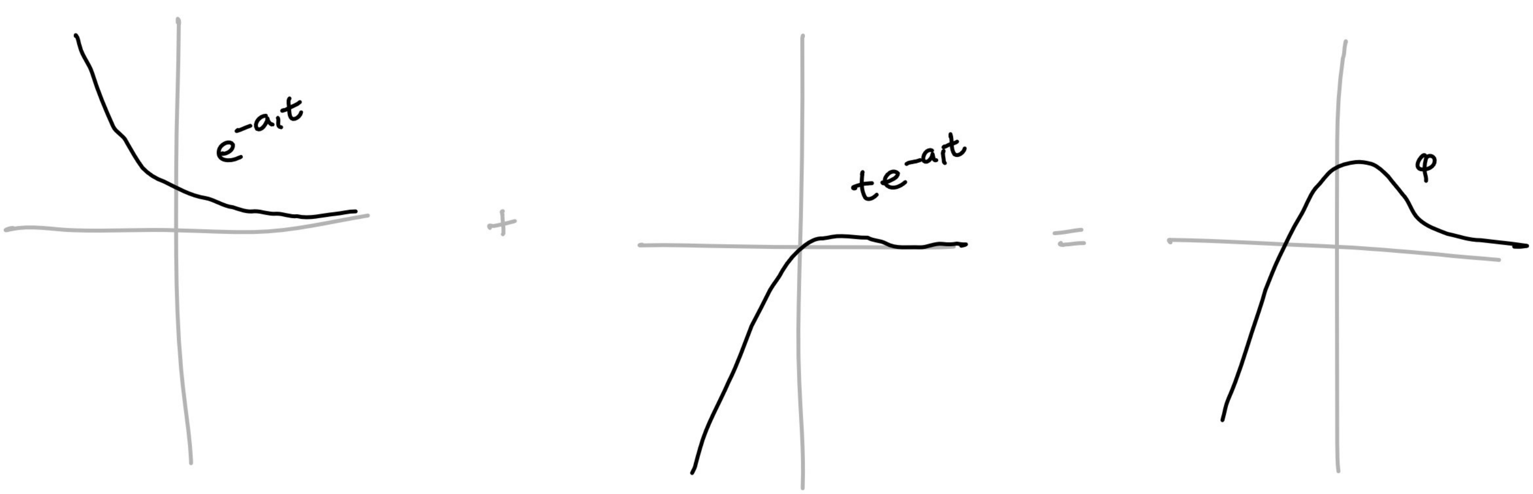


• If $a_1^2 - a_0 = 0$ $\left(\iff a_0 = a_1^2 \right)$,

$\chi(z)$ has a real double root

$$r = -a_1$$

A basis for the space of solutions is $\left\{ e^{-a_1 t}, \; te^{-a_1 t} \right\}$.



Typical solution graph

- If $a_1^2 - a_0 < 0 \quad (\iff a_0 > a_1^2)$

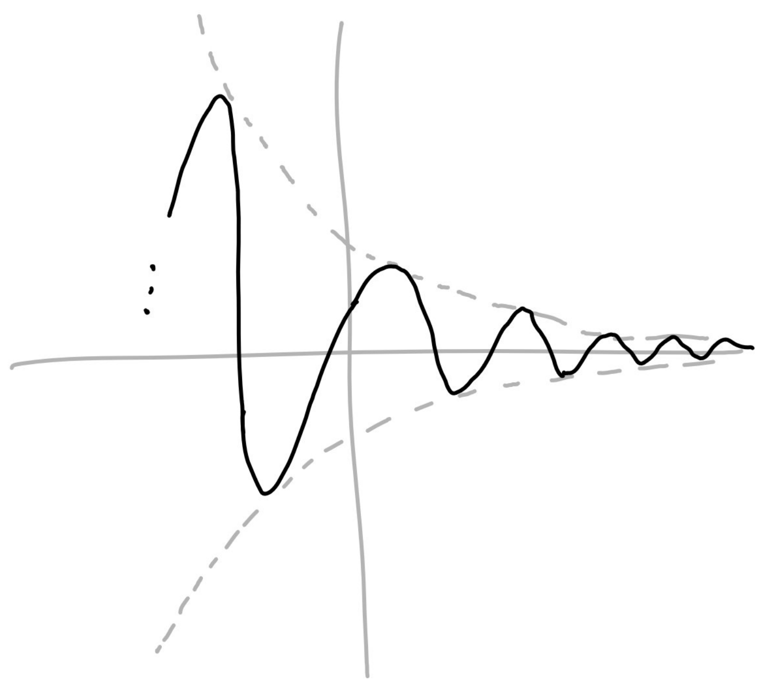
  $\chi(z)$ has two conjugate complex roots

$$r_1 = -a_1 + i\sqrt{a_0 - a_1^2}$$

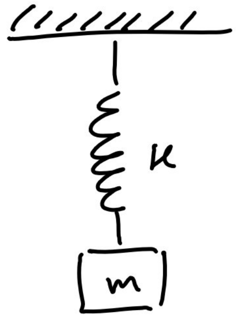
$$r_2 = -a_1 - i\sqrt{a_0 - a_1^2}$$

Basis of space of solutions : $\left\{ e^{-a_1 t} \cos\left(\sqrt{a_0 - a_1^2}\, t\right),\ e^{-a_1 t} \sin\left(\sqrt{a_0 - a_1^2}\, t\right) \right\}$

The graph of a typical solution oscillates between the graphs of $y(t) = e^{-a_1 t}$ and $y(t) = -e^{-a_1 t}$.



let's interpret these solutions in terms of the behaviour of a damped spring-mass system (an example of a damped harmonic oscillator)



We have shown before that the distance of the mass from its equilibrium point is a solution of

$$m \frac{d^2y}{dt^2} + d \frac{dy}{dt} + ky = 0$$

or

$$\frac{d^2y}{dt^2} + \frac{d}{m} \frac{dy}{dt} + \frac{k}{m} y = 0.$$

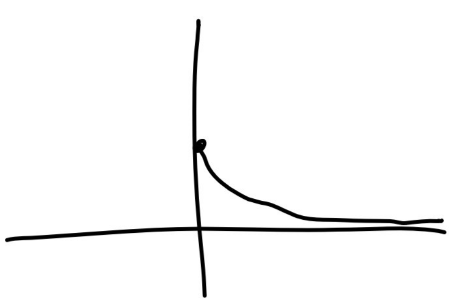
$$2a_1 = \frac{d}{m} \Rightarrow a_1 = \frac{d}{2m}$$

$$a_0 = \frac{k}{m}$$

$$a_1^2 - a_0 = \frac{d^2}{4m^2} - \frac{k}{m}$$

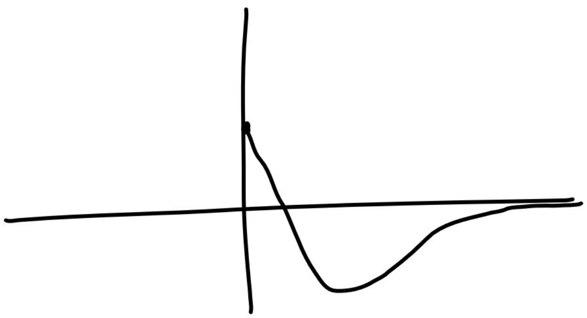
$$= \frac{d^2 - 4km}{4m^2}$$

When $d^2 - 4km > 0$ ($d > 2\sqrt{km}$) : The mass gradually returns to its equilibrium point without oscillation.



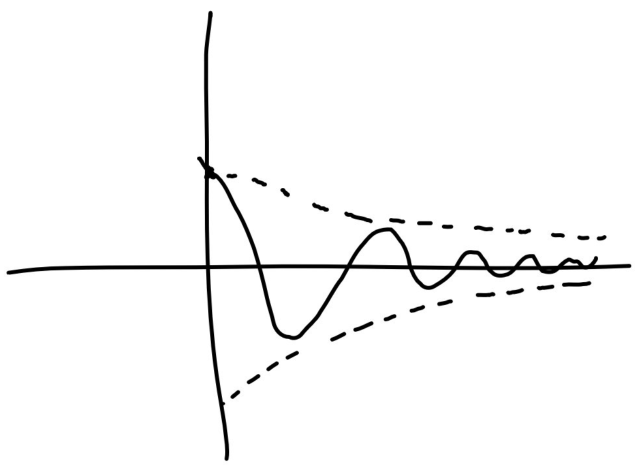
In this case, the spring-mass system is said to be <u>overdamped</u>.

When $d^2 - 4km = 0$ $(d = 2\sqrt{km})$: The mass passes through equilibrium point at most once, then returns to equilibrium.



The spring-mass system is said to be <u>critically damped</u> in this case.

When $d^2 - 4km < 0$ $(d < 2\sqrt{km})$: The mass oscillates about equilibrium, with exponentially decaying amplitude.



The spring-mass system is said to be <u>underdamped</u>