Mthe 237
Lecture 12
Oct. 06, 2017

Topic: General Procedure for Solving Linear Homogeneous Equations with Constant Coefficients

Before we begin, let's establish a useful property of polynomials with real coefficients.

Reminder from last time:

The <u>complex conjugate</u> of a complex number $x+iy$ is

$$\overline{x+iy} = x - iy$$

<u>Examples</u>: $\overline{1+2i} = 1-2i$, $\overline{3} = 3$, $\overline{4i} = -4i$.

Here are two useful observations regarding complex conjugation and the algebraic operations on complex numbers:

$$\begin{aligned}\overline{(x+iy)+(u+iv)} &= \overline{(x+u)+i(y+v)} \\ &= (x+u)-i(y+v) \\ &= (x-iy)+(u-iv) \\ &= \overline{x+iy} + \overline{u+iv}\end{aligned}$$

We can summarize this as:

$$\overline{z+w} = \overline{z} + \overline{w} \quad \text{for all } z, w \in \mathbb{C}.$$

Now, we have

$$\overline{re^{i\theta}} = \overline{r(\cos\theta + i\sin\theta)}$$
$$= \overline{r\cos\theta + ir\sin\theta}$$
$$= r\cos\theta - ir\sin\theta$$
$$= r\cos(-\theta) + ir\sin(-\theta)$$
$$= r(\cos(-\theta) + i\sin(-\theta))$$
$$= re^{-i\theta}$$

This shows the formula
$$\overline{re^{i\theta}} = re^{-i\theta}$$

Now,

$$\overline{(r_1 e^{i\theta_1})(r_2 e^{i\theta_2})} = \overline{r_1 r_2\, e^{i(\theta_1+\theta_2)}}$$
$$= r_1 r_2\, e^{-i(\theta_1+\theta_2)}$$
$$= (r_1 e^{-i\theta_1})(r_2 e^{-i\theta_2})$$
$$= \overline{r_1 e^{i\theta_1}} \cdot \overline{r_2 e^{i\theta_2}}$$

Summarizing, $\overline{zw} = \overline{z}\,\overline{w}$ for all $z, w \in \mathbb{C}$

We could have done this computation in Cartesian coordinates:

$$\overline{(x+iy)(u+iv)} = \overline{(xu-yv) + i(xv+yu)}$$
$$= (xu-yv) - i(xv+yu), \quad \text{whereas}$$

$$\overline{(x+iy)}\ \overline{(u+iv)} = (x-iy)(u-iv)$$
$$= (xu-(-y)(-v)) + i(x(-v) + (-y)u)$$
$$= (xu+yv) - i(xv+yu).$$

Equal

Proposition: If the complex number z is a root of a polynomial with real coefficients,

$$a_r z^r + a_{r-1} z^r + \cdots + a_1 z + a_0 = 0, \qquad a_k \in \mathbb{R}$$

then so is its complex conjugate $\bar{z}$.

Proof: The proof is a computation:

$$\begin{aligned} 0 = \bar{0} &= \overline{a_r z^r + a_{r-1} z^{r-1} + \cdots + a_1 z + a_0} \\ &= \overline{a_r}\,\overline{z^r} + \overline{a_{r-1}}\,\overline{z^{r-1}} + \cdots + \overline{a_1}\,\bar{z} + \overline{a_0} \\ &= a_r \bar{z}^r + a_{r-1} \bar{z}^{r-1} + \cdots + a_1 \bar{z} + a_0 . \end{aligned}$$

This shows $\bar{z}$ is a root. $\square$

Remark. For the above proof, it is important that the coefficients are real numbers.

For polynomials with complex coefficients, the statement of the proposition may not be true.

Example: The two square roots of i are

$$\pm e^{i\pi/4} = \pm \left(\cos\left(\tfrac{\pi}{4}\right) + i \sin\left(\tfrac{\pi}{4}\right) \right) = \pm \frac{1+i}{\sqrt{2}} .$$

So, $$z^2 - i = \left(z + \frac{1+i}{\sqrt{2}} \right)\left(z - \frac{1+i}{\sqrt{2}} \right)$$

$\frac{1+i}{\sqrt{2}}$ and $-\frac{1+i}{\sqrt{2}}$ are not complex conjugates

and we can check, for instance, that

$$\left(\frac{1-i}{\sqrt{2}}\right)^2 - i = \frac{1-2i-1}{2} - i = \frac{-2i}{2} - i = -i-i = -2i \neq 0$$

so the conjugate of the root $\frac{1+i}{\sqrt{2}}$ is <u>not</u> itself a root.

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SOLVING HOMOGENEOUS LINEAR EQUATIONS

<u>Warm-up:</u>

First-order:

$$\frac{dy}{dt} + ry = 0, \quad \text{or} \quad \frac{dy}{dt} = -ry.$$

Can see that $\varphi(t) = e^{-rt}$ is a solution.

First-order linear homogeneous $\Rightarrow$ Solution space is one-dimensional

So $\{e^{-rt}\}$ is a basis of the space of solutions (Clearly e^{-rt} is not the zero function, so $\{e^{-rt}\}$ is linearly independent)

Second-order:

$$(*) \qquad \frac{d^2y}{dt^2} + a_1 \frac{dy}{dt} + a_0 y = 0 \qquad a_1, a_0 \in \mathbb{R}$$

Motivated by the fist-order solution, guess solutions of the form

$$y(t) = e^{rt}, \qquad \text{for sone } r.$$

Plugging into $(*)$, we get

$$r^2 e^{rt} + a_1 r e^{rt} + a_0 e^{rt} = 0$$

$$\iff (r^2 + a_1 r + a_0) e^{rt} = 0$$

$$\iff r^2 + a_1 r + a_0 = 0, \quad \text{as } e^{rt} \neq 0 \text{ for all } t.$$

Let $\chi(z) = z^2 + a_1 z + a_0$ be the <u>characteristic polynomial</u> of $(*)$

khi

Then the above computation shows that e^{rt} is a solution of $(*)$ if and only if r is a root of $\chi(z)$ (meaning $\chi(r) = 0$).

$\chi(z)$ could have two distinct real roots, a real double root, or a pair of conjugate complex roots. Let's look at There possibilities.

a) $\chi(z)$ has two distinct real roots

$$\chi(z) = (z - r_1)(z - r_2), \qquad r_1, r_2 \in \mathbb{R}, \quad r_1 \neq r_2.$$

In this case, $e^{r_1 t}$ and $e^{r_2 t}$ form a basis of the space of solutions (we'll check they are linearly independent later, but we could also compute the Wronskian:

$$\det \begin{pmatrix} e^{r_1 t} & e^{r_2 t} \\ r_1 e^{r_1 t} & r_2 e^{r_2 t} \end{pmatrix} = (r_2 - r_1)\, e^{(r_1 + r_2)t} \neq 0 \text{ if } r_1 \neq r_2 \text{ .)}$$

Basis of solutions: $\boxed{\{ e^{r_1 t}, e^{r_2 t} \}}$

b) $\chi(z)$ has a conjugate pair of complex roots

$$\chi(z) = (z - z_1)(z - \bar{z}_1), \qquad z_1 = \sigma + i\omega \in \mathbb{C} \setminus \mathbb{R}$$

(Note the roots have to be complex conjugates by the first proposition from today, as $\chi(z)$ has real coefficients.)

$e^{(\sigma + i\omega)t}$ and $e^{(\sigma - i\omega)t}$ is then a $\mathbb{C}$-basis for complex-valued solutions.

By the principle of superposition,

$$\frac{e^{(\sigma+i\omega)t} + e^{(\sigma-i\omega)t}}{2} = \frac{e^{\sigma t} e^{i\omega t} + e^{\sigma t} e^{-i\omega t}}{2}$$

$$= e^{\sigma t}\left(\frac{e^{i\omega t} + e^{-i\omega t}}{2}\right)$$

$$= e^{\sigma t} \cos(\omega t)$$

and

$$\frac{e^{(\sigma+i\omega)t} - e^{(\sigma-i\omega)t}}{2i} = \frac{e^{\sigma t} e^{i\omega t} - e^{\sigma t} e^{-i\omega t}}{2i}$$

$$= e^{\sigma t}\left(\frac{e^{i\omega t} - e^{-i\omega t}}{2i}\right)$$

$$= e^{\sigma t} \sin(\omega t)$$

are again solutions. These are real-valued and linearly-independent (to be proved in future, or follow from

$$\det\begin{pmatrix} e^{\sigma t}\cos(\omega t) & e^{\sigma t}\sin(\omega t) \\ \sigma e^{\sigma t}\cos(\omega t) - \omega e^{\sigma t}\sin(\omega t) & \sigma e^{\sigma t}\sin(\omega t) + \omega e^{\sigma t}\cos(\omega t) \end{pmatrix}$$

$$= e^{2\sigma t}\Big(\sigma\cos(\omega t)\sin(\omega t) + \omega\cos^2(\omega t) - \big(\sigma\cos(\omega t)\sin(\omega t) - \omega\sin^2(\omega t)\big)\Big)$$

$$= e^{2\sigma t}\left(\omega \cos^2(\omega t) + \omega \sin^2(\omega t) \right) = \omega e^{2\sigma t}$$

$\neq 0$ because $z_1 = \sigma + i\omega$ isn't real

Basis of Solutions: $\boxed{\{ e^{\sigma t}\cos(\omega t),\ e^{\sigma t}\sin(\omega t) \}}$

c) $\chi(z)$ has a repeated real root

$$\chi(z) = (z-r)^2, \qquad r \in \mathbb{R}.$$

In this case, e^{rt} is a solution, but we need to find another linearly independent solution. It turns out that

te^{rt} works

Check: $$\frac{d}{dt}(te^{rt}) = e^{rt} + rte^{rt} = (1+rt)e^{rt}$$

$$\frac{d^2}{dt^2}(te^{rt}) = re^{rt} + (1+rt)re^{rt}$$
$$= (2r + r^2 t)e^{rt}$$

$$\frac{d^2}{dt^2}(te^{rt}) + a_1 \frac{d}{dt}(te^{rt}) + a_0(te^{rt})$$
$$= \left((2r + r^2 t) + a_1(1+rt) + a_0 t \right) e^{rt}$$
$$= \left(t(r^2 + a_1 r + a_0) + 2r + a_1 \right) e^{rt}$$

Now, $\chi(r) = r^2 + a_1 r + a_0 = 0$ by hypothesis that r is a root of $\chi(z)$,

and $\chi(z) = z^2 + a_1 z + a_0$

$$= (z-r)^2 = z^2 - 2rz + r^2$$

so $a_1 = -2r$ by comparing coefficients.

Therefore,

$$\left(t(r^2 + a_1 r + a_0) + 2r + a_1 \right) = 0$$

and te^{rt} is a solution.

Will check later that e^{rt} and te^{rt} are linearly independent

(or, $\det \begin{pmatrix} e^{rt} & te^{rt} \\ re^{rt} & e^{rt} + rte^{rt} \end{pmatrix}$

$= e^{2rt} + rte^{2rt} - rte^{2rt} = e^{2rt} \neq 0$)

Basis of Solutions: $\boxed{\{e^{rt}, te^{rt}\}}$

Generally,

$$(*) \qquad \frac{d^r y}{dt^r} + a_{r-1}\frac{d^{r-1} y}{dt^{r-1}} + \cdots + a_1 \frac{dy}{dt} + a_0 y = 0$$

$$a_k \in \mathbb{R}$$

Write down the characteristic polynomial of $(*)$:

$$\chi(z) = z^r + a_{r-1} z^{r-1} + \cdots + a_1 z + a_0$$

Factor $\chi(z)$ over the complex numbers:

$$\chi(z) = (z-r_1)^{m_1} \cdots (z-r_k)^{m_k} \left[(z-z_1)(z-\bar{z}_1)\right]^{n_1} \cdots \left[(z-z_\ell)(z-\bar{z}_\ell)\right]^{n_\ell}$$

Where $r_1, \ldots, r_k$ are real and pairwise distinct

$z_1, \ldots, z_\ell$ are complex and pairwise distinct

Write $z_k = \sigma_k + i\omega_k$, $\sigma_k, \omega_k \neq 0$ real.

For each real root r_j, take

$$\boxed{e^{r_j t},\ t e^{r_j t},\ t^2 e^{r_j t},\ \ldots,\ t^{m_j - 1} e^{r_j t}}$$

as basis elements for the space of sols.

For each complex root z_j, take

$$\boxed{\begin{aligned} &e^{\sigma_j t}\cos(\omega_j t),\ t e^{\sigma_j t}\cos(\omega_j t), \ldots,\ t^{n_j - 1} e^{\sigma_j t}\cos(\omega_j t), \\ &e^{\sigma_j t}\sin(\omega_j t),\ t e^{\sigma_j t}\sin(\omega_j t), \ldots,\ t^{n_j - 1} e^{\sigma_j t}\sin(\omega_j t) \end{aligned}}$$

as basis elements for the space of sols.

Counting, we get $\deg(\chi(z)) = \text{order}(*)$ basis elements, hence have found a basis.

Up to factoring $\chi(z)$ (which is always possible in principle by the fundamental theorem of algebra), this is a complete solution to any linear homogeneous equation with constant coefficients.

Examples:

$$\frac{d^4y}{dt^4} + 2\frac{d^3y}{dt^3} + 6\frac{d^2y}{dt^2} - 22\frac{dy}{dt} + 13y = 0$$

$$\chi(z) = z^4 + 2z^3 + 6z^2 - 22z + 13$$
$$= (z-1)^2\left[(z-(-2+3i))(z-(-2-3i))\right]$$

Roots: $r_1 = 1$, $m_1 = 2$

$z_1 = -2+3i$, $n_1 = 1$

Basis of solutions:

$$\underbrace{e^t,\ te^t}_{\text{From double real root}},\ \underbrace{e^{-2t}\cos(3t),\ e^{-2t}\sin(3t)}_{\text{From conjugate pair of complex roots}}$$

$$\frac{d^4y}{dt^4} - 4\frac{d^3y}{dt^3} + 14\frac{d^2y}{dt^2} - 20\frac{dy}{dt} + 25y = 0$$

$$\chi(z) = z^4 - 4z^3 + 14z^2 - 20z + 25$$

$$= \left[(z-(1+2i))(z-(1-2i))\right]^2$$

$z_1 = 1+2i$, $n_1 = 2$

Basis of solutions:

$e^t\cos(2t)$, $te^t\cos(2t)$, $e^t\sin(2t)$, $te^t\sin(2t)$.