MTHE 237
Lecture 11
Oct. 04, 2017

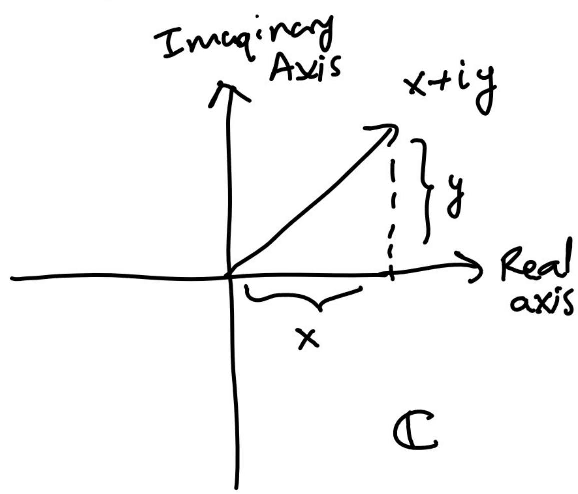
Topic: Introduction to $\mathbb{C}$ (complex numbers)

Def. A <u>complex number</u> is an ordered pair (x,y), where x and y are real numbers, called the <u>real</u> and <u>imaginary</u> parts of (x,y), respectively.

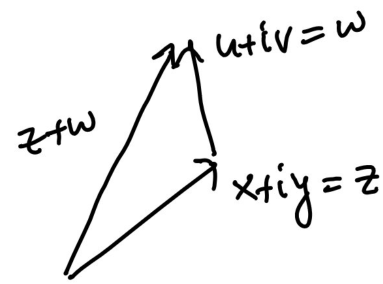
Informally, we write (x,y) as $x+iy$, where i is the <u>imaginary unit</u>, a symbol with the property that $i^2=-1$.

Other common notations for i are $\sqrt{-1}$ (e.g. when one wants to reserve the use of i for indices) or j (especially in electrical engineering, where i is reserved for current).

Geometrically, $x+iy$ can be thought of as a 2-dimensional vector. With this point of view, the set $\mathbb{C}$ of complex numbers is called the <u>complex plane</u>.



§ Arithmetic and other basic operations with complex numbers:

Operation	Geometric Interpretation
Addition $(x+iy)+(u+iv)$ $= (x+u)+i(y+v)$	Vector addition 
Multiplication $(x+iy)(u+iv)$ $= (xu-yv)+i(xv+yu)$	Please see below – best understood in polar representation
Conjugation $\overline{x+iy} = x-iy$	Reflection about the real axis (that is, the x-axis)
Norm or Magnitude $\lvert x+iy \rvert = \sqrt{x^2+y^2}$ $= \sqrt{(x+iy)(\overline{x+iy})}$	Vector length

Examples: $(1+2i)+(3+4i) = (1+3)+i(2+4)$

$$= 4+6i$$

$$(1+2i)(3+4i) = (3-8)+i(4+6)$$

$$= -5+10i$$

$$\overline{1+2i} = 1-2i$$

$$|3+4i| = \sqrt{3^2+4^2} = 5$$

Historically, complex numbers were introduced in the study of roots of polynomials. (Reminder: a number r is called a <u>root</u> of a polynomial $p(x)$ if $p(r)=0$.)

Already the degree 2 polynomial

$$ax^2+bx+c \quad , \quad a, b, c \in \mathbb{R}$$

has complex number roots when the discriminant $b^2-4ac<0$

(As you remember, the roots are given by the quadratic equation $\dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$.)

Even more intriguingly, in the analogous cubic equation, complex numbers are necessary to express even real solutions.

Once one allows complex roots, the problem of finding the roots of a polynomial is completely solved. This is the content of:

Theorem (Fundamental Theorem of Algebra). Every nonconstant polynomial with complex coefficients,

$$p(z) = a_r z^r + a_{r-1} z^{r-1} + \dots + a_1 z + a_0 \quad , \quad a_k \in \mathbb{C},$$

has a complex root.

Corollary. Every polynomial with complex coefficients factors into a product of linear terms:

$$p(z) = a(z - w_1) \cdots (z - w_r), \quad \text{where } w_k \in \mathbb{C}$$

Proof sketch: If $p(z)$ is constant, $p(z) = a$, there is nothing to prove ($p(z)$ is already in the claimed form).

Similarly, if $p(z)$ has degree 1, then

$$p(z) = a_1 z + a_0 = a_1\left(z + \frac{a_0}{a_1}\right).$$

If $p(z)$ has degree $r > 1$, by the Fundamental Theorem of Algebra $p(w_r) = 0$ for some $w_r \in \mathbb{C}$.

Therefore, $(z - w_r)$ is a factor of $p(z)$; that is, we have $p(z) = (z - w_r)g(z)$ for some polynomial $g(z)$

[This is not a completely justified statement and is the reason this is only a proof sketch.]

Then, $g(z) = \dfrac{p(z)}{z - w_r}$ is a polynomial of degree lower than $p(z)$.

Repeating this procedure at most finitely many times (at most finitely many because degree goes down each time), we get

$$\frac{p(z)}{(z-w_2)\cdots(z-w_r)} = a_1 z + a_0 = a_1\left(z + \frac{a_0}{a_1}\right),$$

so that multiplying both sides by $(z-w_2)\cdots(z-w_r)$,

$$p(z) = a_1\left(z + \frac{a_0}{a_1}\right)(z-w_2)\cdots(z-w_r),$$

which has the desired form. $\blacksquare$

§ Complex Exponential

Q: What should be the meaning of e^z, $z \in \mathbb{C}$?

There are a few natural requirements we might make:

1) $e^{z+w} = e^z e^w$ for all $z, w \in \mathbb{C}$ (Exponent law)

2) $\frac{d}{dz} e^z = e^z$ (whatever $\frac{d}{dz}$ might mean...)

3) $e^z = 1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \cdots + \frac{z^n}{n!} + \cdots$

Notice that 3) implies that when $z = x + i0$ (so that z is purely real), we recover the usual real exponential, which has the power series

$$e^x = 1 + x + \frac{x^2}{2} + \cdots + \frac{x^n}{n!} + \cdots$$

3) also implies 1) and 2), so this is not a logically minimal list.

By 1), writing $z = x + iy$, we have

$$e^z = e^x e^{iy}$$

Already has a meaning

So we reduced to defining e^{iy} for y real.

We change notation to $e^{i\theta}$ from e^{iy}.

<u>Aside</u>: A $\mathbb{C}$-valued function of a real variable is given by two coordinate functions,

$$f : t \mapsto x(t) + i y(t).$$

This may be thought of as a parametrization of a curve in $\mathbb{C}$.

The derivative of a $\mathbb{C}$-valued function with respect to t is taken coordinatewise

$$\frac{df}{dt} : t \mapsto \frac{dx}{dt} + i \frac{dy}{dt}$$

If we are thinking of f as a parametrization, the derivative is the velocity of the parametrization.

The theory we developed for linear homogeneous equations extends nearly verbatim to equations of the form

$$(*) \qquad \frac{d^r y}{dt} + a_{r-1}(t) \frac{d^{r-1} y}{dt^{r-1}} + \dots + a_1(t) \frac{dy}{dt} + a_0(t) y = 0,$$

where $a_k : I \to \mathbb{C}$ are $\mathbb{C}$-valued coefficient functions (solutions of $(*)$ are then also $\mathbb{C}$-valued; they are functions $\varphi : I \to \mathbb{C}$)

The Existence and Uniqueness Theorem holds in this setting, and implies that the set of solutions of $(*)$ is a complex vector space. (End of Aside)

Similar to $\frac{d}{dz} e^z = e^z$, we should require that

$$\frac{d}{d\theta} e^{i\theta} = i e^{i\theta} \quad \text{and} \quad \frac{d^2}{d\theta^2} e^{i\theta} = i^2 e^{i\theta} = -e^{i\theta}.$$

If this is true, the function $e^{i\theta}$ is a solution of the differential equation

$$\frac{d^2 y}{d\theta^2} + y = 0.$$

We checked before that $\{\sin(\theta), \cos(\theta)\}$ is a fundamental set of solutions (a basis of solution space).

Therefore, we have

$$e^{i\theta} = A\cos\theta + B\sin\theta \quad \text{for some } A, B \in \mathbb{C}$$

What should the initial conditions be?

$$e^{i\theta}\Big|_{\theta=0} = 1,$$

$$\frac{d}{d\theta} e^{i\theta}\Big|_{\theta=0} = i.$$

$$1 = A\cos(0) + B\sin(0) = A$$

$$i = -A\sin(0) + B\cos(0) = B$$

So that, finally, $\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$

This is called <u>Euler's identity</u>.

We have

$$\frac{e^{i\theta}+e^{-i\theta}}{2} = \frac{(\cos\theta + i\sin\theta) + (\cos(-\theta) + i\sin(-\theta))}{2}$$

$$= \frac{(\cos\theta + i\sin\theta) + (\cos\theta - i\sin\theta)}{2}$$

Since cos is an even function and sin is an odd function

$$= \frac{(\cos\theta + \cos\theta) + i(\sin\theta - \sin\theta)}{2}$$

$$= \cos\theta$$

$$\frac{e^{i\theta}-e^{-i\theta}}{2i} = \frac{(\cos\theta + i\sin\theta) - (\cos(-\theta) + i\sin(-\theta))}{2i}$$

$$= \frac{(\cos\theta + i\sin\theta) - (\cos\theta - i\sin\theta)}{2i}$$

$$= \frac{(\cos\theta - \cos\theta) + i(\sin\theta + \sin\theta)}{2i}$$

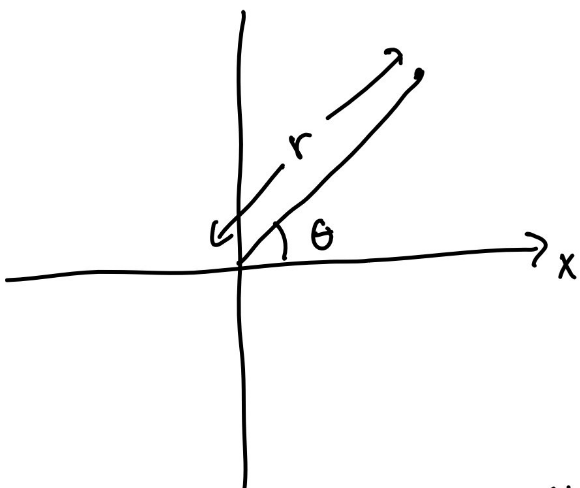
$$= \sin\theta.$$

These computations show the identities

$$\boxed{\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}, \qquad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}}$$

§ Polar representation of Complex Numbers

This is simply polar coordinates for complex numbers.



A complex number may be specified by the length of the line segment connecting it to the origin, and the angle that line segment makes with the real axis.

We write $z = re^{i\theta}$, $\left.\begin{array}{l} r \geq 0 \\ \theta \in [0, 2\pi) \end{array}\right\}$ Both r and θ are real numbers

Polar → Cartesian Representation:

$z = re^{i\theta}$

$$\frac{x}{r} = \cos\theta \implies x = r\cos\theta$$

$$\frac{y}{r} = \sin\theta \implies y = r\sin\theta$$

Cartesian → Polar Representation:

$z = x + iy$

$$r = \sqrt{x^2 + y^2} = |z|$$

$$\theta \text{ "=" } \arctan\left(\frac{y}{x}\right)$$

Polar form lets us interpret multiplication of complex numbers geometrically:

$$z = r_1 e^{i\theta_1}, \qquad w = r_2 e^{i\theta_2}$$

$$zw = \left(r_1 e^{i\theta_1}\right)\left(r_2 e^{i\theta_2}\right)$$

$$= \underbrace{r_1 r_2}_{\text{Lengths get multiplied}} e^{\overbrace{i(\theta_1+\theta_2)}^{}}$$

$$\underbrace{}_{\text{Angles get added}}$$

<u>Example</u>:

$$z = 2e^{i\pi/4}$$

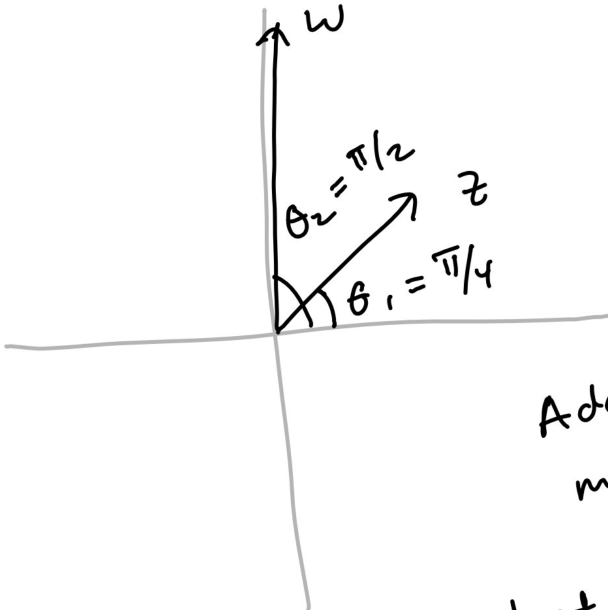
$$= 2\left(\cos\left(\tfrac{\pi}{4}\right) + i\sin\left(\tfrac{\pi}{4}\right)\right)$$

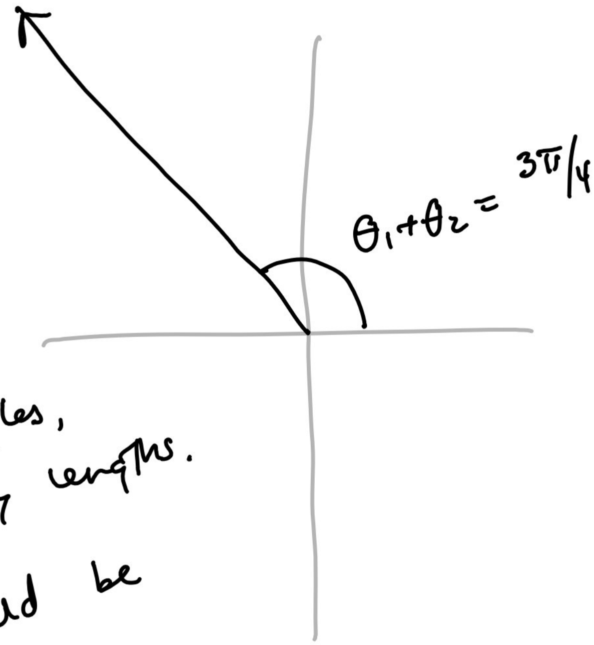
$$= 2\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) = \sqrt{2} + i\sqrt{2}$$

$$w = 3e^{i\pi/2}$$

$$= 3\left(\cos\left(\tfrac{\pi}{2}\right) + i\sin\left(\tfrac{\pi}{2}\right)\right)$$

$$= 3i$$





Add angles,
multiply lengths.

Product should be

$6e^{i3\pi/4}$

$= 6\left(\cos(3\pi/4) + i\sin\left(\frac{3\pi}{4}\right)\right)$

$= -\frac{6}{\sqrt{2}} + i\frac{6}{\sqrt{2}} = -3\sqrt{2} + 3i\sqrt{2}$

Can check this using the previous def. of multiplication:

$$(\sqrt{2} + \sqrt{2}i)(3i) = 3i\sqrt{2} + 3\sqrt{2}i^2$$

$$= -3\sqrt{2} + 3i\sqrt{2}, \text{ as expected.}$$

§ Finding Complex Roots

Finally, here is a brief explanation of how to find n^{th} complex roots.

First, let's discuss the problem of factoring

$$z^n - 1, \quad n \text{ a positive integer.}$$

The roots are called the n^{th} roots of unity.

Writing $z = re^{i\theta}$, we have

$$(re^{i\theta})^n = r^n e^{in\theta} = 1$$

This means $r = 1$ (since $r \geq 0$ and real) and

$n\theta = 2\pi k$ for some $k \in \mathbb{Z}$.

$$\theta = \frac{2\pi k}{n}.$$

Since θ is 2π-periodic, this gives n different values of θ:

$$0, \frac{2\pi}{n}, \frac{4\pi}{n}, \dots, \frac{2\pi k}{n}, \dots, \frac{2\pi(n-1)}{n}, \text{ hence}$$

n roots:

$$1, e^{i\frac{2\pi}{n}}, e^{i\frac{4\pi}{n}}, \dots, e^{i\frac{2\pi k}{n}}, \dots, e^{i\frac{2\pi(n-1)}{n}}.$$

Since $z^n - 1$ has degree n, it has at most n roots, so these are all of them.

Now, if w is any complex number, write w in polar representation as

$$w = re^{i\theta}$$

Then, $z = r^{1/n} e^{i\theta/n}$ is clearly a complex number satisfying $z^n = w$, that is, an n^{th} root of w.

Moreover, multiplying z by any of the n^{th} roots of unity, $e^{\frac{2\pi k}{n} i}$, we have

$$\left(r^{1/n} e^{i\theta/n} e^{\frac{2\pi k}{n} i}\right)^n = re^{i\theta} e^{2\pi k i}$$

$$= re^{i\theta}$$

So $z \cdot e^{i\frac{2\pi k}{n}} = r^{1/n} e^{i\theta/n} e^{i\frac{2\pi k}{n}}$ is another n^{th} root of w.

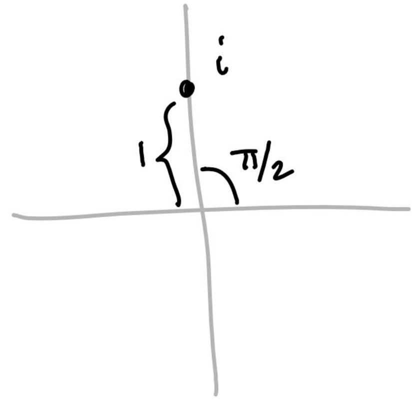
Since by definition n^{th} roots of w are the roots of the polynomial $z^n - w = 0$,
by degree reasons we found all of the n^{th} roots of w. They are

$$\left(r^{1/n} e^{i\frac{\theta}{n}}\right) e^{i\frac{2\pi k}{n}}, \quad k = 0, 1, 2, \dots, n-1.$$

Example: What are all of the sixth roots of i?

In polar form, $|i| = \sqrt{0^2+1^2} = 1$, so

$$i = e^{i\frac{\pi}{2}}$$



The sixth roots of unity are:

$$1,\ e^{i\frac{2\pi}{6}},\ e^{i\frac{4\pi}{6}},\ e^{i\frac{6\pi}{6}},\ e^{i\frac{8\pi}{6}},\ e^{i\frac{10\pi}{6}}.$$

Therefore, the sixth roots of i are:

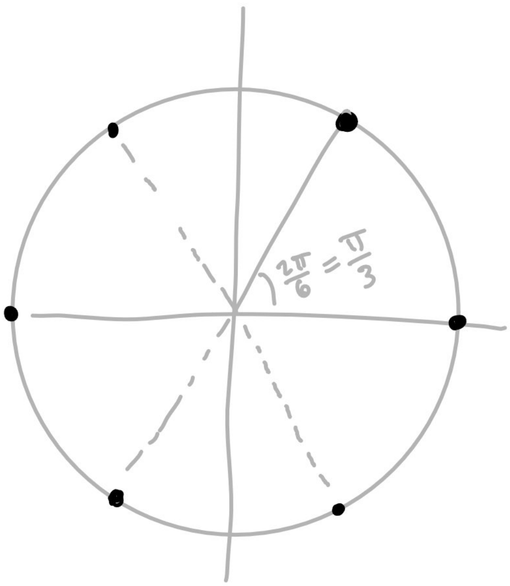
$$e^{i\frac{\pi}{12}},\ e^{i\left(\frac{\pi}{12}+\frac{2\pi}{6}\right)},\ e^{i\left(\frac{\pi}{12}+\frac{4\pi}{6}\right)},\ \ldots,\ e^{i\left(\frac{\pi}{12}+\frac{10\pi}{6}\right)}$$

or

$$e^{i\frac{\pi}{12}},\ e^{i\frac{5\pi}{12}},\ e^{i\frac{9\pi}{12}},\ e^{i\frac{13\pi}{12}},\ e^{i\frac{17\pi}{12}},\ e^{i\frac{21\pi}{12}}.$$

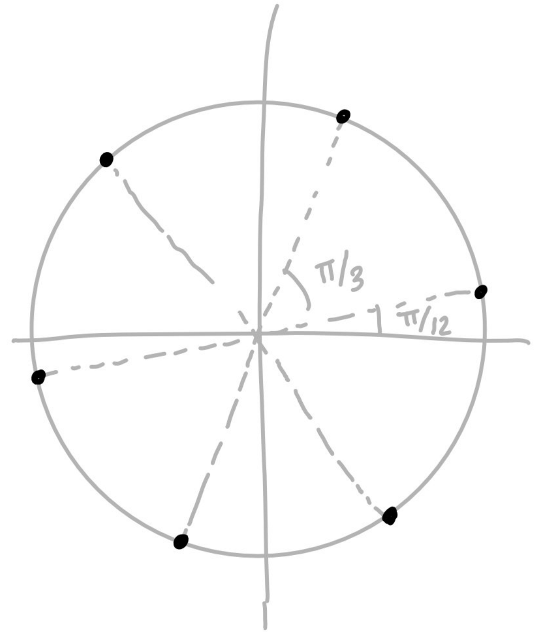
To put these into form $x+iy$, we can apply Euler's formula: for instance,

$$e^{i\frac{9\pi}{12}} = \cos\left(\frac{9\pi}{12}\right) + i\sin\left(\frac{9\pi}{12}\right).$$



The sixth roots of unity

(They are vertices of a regular hexagon; all roots of unity are vertices of regular polygons.)



The sixth roots of i

(The picture is rotated by $\frac{\pi}{12}$)