Mthe 237
Lecture 10
Oct. 03, 2017

Topic: Wronskians and Linear Independence in $C^r(I, \mathbb{R})$

Preliminaries:

Lin. Algebra: A matrix $A = \begin{pmatrix} a_{11} & \cdots & a_{1r} \\ \vdots & \ddots & \vdots \\ a_{r1} & \cdots & a_{rr} \end{pmatrix}$ defines a linear map $L : \mathbb{R}^r \to \mathbb{R}^r$

$$\begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_r \end{pmatrix} \mapsto A \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_r \end{pmatrix} = \begin{pmatrix} a_{11}\alpha_1 + \cdots + a_{1r}\alpha_r \\ \vdots \\ a_{r1}\alpha_1 + \cdots + a_{rr}\alpha_r \end{pmatrix}$$

By choosing a basis for a vector space, we are choosing an isomorphism with $\mathbb{R}^r$ for some r. Every linear map has a corresponding matrix after a choice of basis.

The determinant of a matrix determines whether the associated linear map is invertible.

$\det A = 0 \iff$ Map associated to A is not invertible

$\iff$ Map associated to A has a nonzero kernel.

(There is a nonzero vector v with $Av = 0$.)

Logic: In logic, it is necessary to make a distinction between the statements

A implies B ($A \Rightarrow B$) and

B implies A ($B \Rightarrow A$)

Example: A : The person is Newton.

B : The person is a man.

$A \Rightarrow B$: If the person is Newton, then the person is a man.

is **True**.

$B \Rightarrow A$: If the person is a man, then the person is Newton.

is **False** (the person could be Leibniz, the mortal enemy of Newton)

The statement $A \Rightarrow B$ is logically equivalent to $(\text{not } B) \Rightarrow (\text{not } A)$, called its **contrapositive**

In the example: "If the person is not a man, then the person is not Newton." is the contrapositive.

The statement (not A) $\Rightarrow$ (not B) is logically independent of $A \Rightarrow B$; it is the contrapositive of $B \Rightarrow A$.

In the example: "If the person is not Newton, then he is not a man."

Is (not A) $\Rightarrow$ (not B).

This is false (and equivalent to $B \Rightarrow A$).

§

__Def.__ For $f_1, \dots, f_r \in C^{r-1}(I, \mathbb{R})$, their __Wronskian__ is the function

$$W(f_1, \dots, f_r)(t) = \det \begin{pmatrix} f_1(t) & \cdots & f_r(t) \\ \frac{df_1}{dt}(t) & & \frac{df_r}{dt}(t) \\ \vdots & \ddots & \vdots \\ \frac{d^{r-1}f_1}{dt^{r-1}}(t) & \cdots & \frac{d^{r-1}f_r}{dt^{r-1}}(t) \end{pmatrix}$$

with domain I.

__Example:__ $f_1(t) = t$, $f_2(t) = t^2$ in $\underbrace{C^\infty(I, \mathbb{R})}$

__Def.__ $C^\infty(I, \mathbb{R}) = \bigcap_r C^r(I, \mathbb{R})$

$$W(t, t^2)(t) = \det \begin{pmatrix} t & t^2 \\ 1 & 2t \end{pmatrix}$$

$$= 2t^2 - t^2 = t^2.$$

Proposition: Let $f_1, \dots, f_r \in C^{r-1}(I, \mathbb{R})$.

If there exists a $t_0 \in I$ such that

$$W(f_1, \dots, f_r)(t_0) \neq 0,$$

then $f_1, \dots, f_r$ are linearly independent.

Proof:

We prove the contrapositive:

A: There exists a $t_0 \in I$ with $W(t_0) \neq 0$

not A: For all $t \in I$, $W(t) = 0$

B: $f_1, \dots, f_r$ are linearly independent

not B: $f_1, \dots, f_r$ are linearly dependent.

$A \Rightarrow B$ is equivalent to $(\text{not } B) \Rightarrow (\text{not } A)$

If $f_1, \dots, f_r$ are linearly dependent, then $W(f_1, \dots, f_r)(t) = 0$ for all $t \in I$.

Suppose that $f_1, \dots, f_r$ are linearly dependent, so that there exist $\alpha_1, \dots, \alpha_r$ not all zero with

$$(*) \quad \alpha_1 f_1 + \dots + \alpha_r f_r = 0$$

(This means $\alpha_1 f_1(t) + \dots + \alpha_r f_r(t) = 0$ for all $t \in I$)

Differentiating $(*)$ $(r-1)$ times,

$$\alpha_1 \frac{df_1}{dt}(t) + \cdots + \alpha_r \frac{df_r}{dt}(t) = 0,$$

$$\alpha_1 \frac{d^2 f_1}{dt^2}(t) + \cdots + \alpha_r \frac{d^2 f_r}{dt^2}(t) = 0,$$

$$\vdots$$

$$\alpha_1 \frac{d^{r-1} f_1}{dt^{r-1}}(t) + \cdots + \alpha_r \frac{d^{r-1} f_r}{dt^{r-1}}(t) = 0.$$

This may be rewritten as the matrix equation

$$\begin{pmatrix} f_1(t) & \cdots & f_r(t) \\ \vdots & \ddots & \vdots \\ \frac{d^{r-1} f_1}{dt^{r-1}}(t) & \cdots & \frac{d^{r-1} f_r}{dt^{r-1}}(t) \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_r \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

for all $t \in I$

So the matrix

$$\begin{pmatrix} f_1(t) & \cdots & f_r(t) \\ \vdots & \ddots & \vdots \\ \frac{d^{r-1} f_1}{dt^{r-1}}(t) & \cdots & \frac{d^{r-1} f_r}{dt^{r-1}}(t) \end{pmatrix}$$

has a nonzero element in its kernel for all $t \in I$. This implies that its determinant, $W(f_1, \ldots, f_r)(t)$, is zero for all $t \in I$. $\blacksquare$

Example: $W(t, t^2)(1) = 1^2 = 1 \neq 0$, so t, t^2 are linearly independent.

Q: Is the converse true?

If $f_1, \dots, f_r$ are linearly independent, is it true that there exists some $t_0 \in I$ with

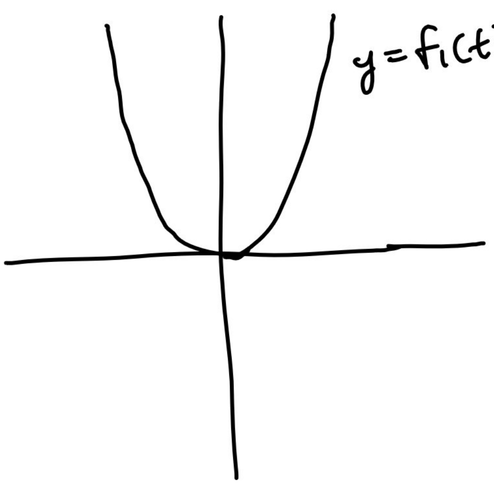
$$W(f_1, \dots, f_r)(t_0) \neq 0 ?$$

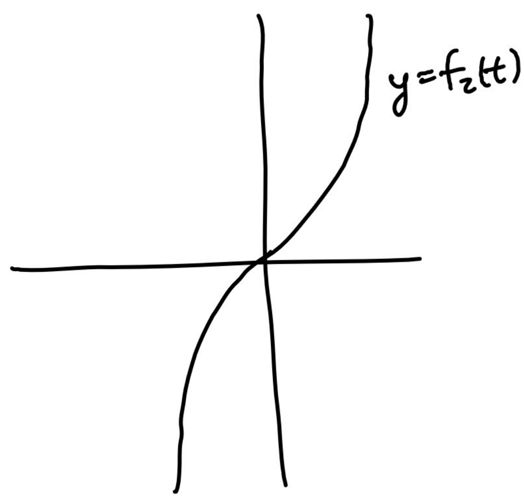
A: In general, no.

Example (Peano)

Let

$$f_1(t) = t^2 \qquad f_2(t) = t|t| = \begin{cases} t^2, & t \geq 0 \\ -t^2, & t < 0 \end{cases}$$





$$\frac{df_1}{dt}(t) = 2t$$

$$\frac{df_2}{dt}(t) = 2t \quad \text{for } t>0$$

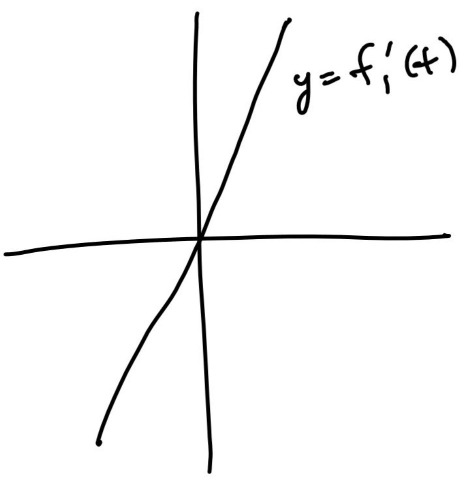
$$\frac{df_2}{dt}(t) = -2t \quad \text{for } t<0$$

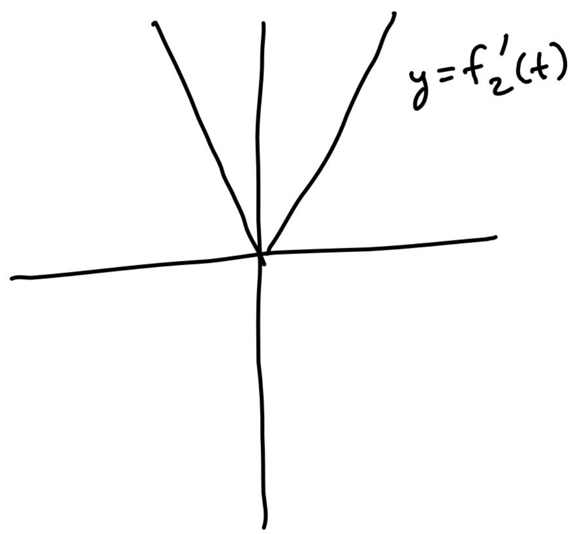
$$\frac{df_2}{dt}(0) = \lim_{h\to 0} \frac{f_2(0+h) - f_2(0)}{h}$$

$$= \lim_{h\to 0} \frac{h|h| - 0}{h}$$

$$= \lim_{h\to 0} |h| = 0$$

$$\frac{df_2}{dt}(t) = 2|t|$$





We have $f_1, f_2 \in C^1(\mathbb{R}, \mathbb{R})$

$$W(t^2, t|t|)(t) = \det \begin{pmatrix} t^2 & t|t| \\ 2t & 2|t| \end{pmatrix} = 2t^2|t| - 2t^2|t| = 0$$

for all $t \in \mathbb{R}$.

However, f_1 and f_2 are linearly independent:

Suppose

$$\alpha_1 t^2 + \alpha_2 t|t| = 0.$$

At $t=1$: $\alpha_1 \cdot 1 + \alpha_2 \cdot 1 = 0$

$t=-1$: $\alpha_1 \cdot 1 + \alpha_2 \cdot (-1) = 0$

Obtain the system of equations

$$\left.\begin{array}{l} \alpha_1 + \alpha_2 = 0 \\ \alpha_1 - \alpha_2 = 0 \end{array}\right\}$$

Adding the first to the second, $2\alpha_1 = 0$,
so $\alpha_1 = 0$ and $\alpha_2 = 0$.

However, the converse <u>is</u> true if $f_1, \dots, f_r$ is a fundamental set of solutions of a homogeneous linear diff. eq.!

<u>Proposition</u>. Let $\varphi_1, \dots, \varphi_r$ be solutions of

$$\frac{d^r y}{dt^r} + a_{r-1}(t) \frac{d^{r-1} y}{dt^{r-1}} + \cdots + a_1(t) \frac{dy}{dt} + a_0(t) y = 0,$$

over $I \subset \mathbb{R}$, where $a_{r-1}, \dots, a_1, a_0$ are continuous over I

The following are equivalent:

i) $\varphi_1, \dots, \varphi_r$ are linearly independent

ii) There exists $t_0 \in I$ with $W(\varphi_1, \dots, \varphi_r)(t_0) \neq 0$.

<u>Proof</u>: ii) $\Rightarrow$ i) was the first proposition today.

i) $\Rightarrow$ ii):

We show the contrapositive:

> If $W(\varphi_1, \dots, \varphi_r)(t) = 0$ for all $t \in I$, then $\varphi_1, \dots, \varphi_r$ are linearly dependent.

If $W(\varphi_1, \dots, \varphi_r)(t) = 0$ for all t, in particular $W(\varphi_1, \dots, \varphi_r)(t_0) = 0$ for some t_0.

$$\det \begin{pmatrix} \varphi_1(t_0) & \cdots & \varphi_r(t_0) \\ \vdots & \ddots & \vdots \\ \frac{d^{r-1}\varphi_1}{dt^{r-1}}(t_0) & \cdots & \frac{d^{r-1}\varphi_r}{dt^{r-1}}(t_0) \end{pmatrix} = 0$$

There exists a nonzero vector $\begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_r \end{pmatrix}$ in the kernel of the matrix, so that

$$(*) \qquad \begin{pmatrix} \varphi_1(t_0) & \cdots & \varphi_r(t_0) \\ \vdots & \ddots & \vdots \\ \frac{d^{r-1}\varphi_1}{dt^{r-1}}(t_0) & \cdots & \frac{d^{r-1}\varphi_r}{dt^{r-1}}(t_0) \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_r \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

Let $g = \alpha_1 \varphi_1 + \cdots + \alpha_r \varphi_r$. g is a solution of the linear eq., since the set of solutions is a vector space.

Writing out $(*)$, we have

$$\alpha_1 \varphi_1(t_0) + \cdots + \alpha_r \varphi_r(t_0) = 0$$

or $g(t_0) = 0$

$$\alpha_1 \frac{d\varphi_1}{dt}(t_0) + \cdots + \alpha_r \frac{d\varphi_r}{dt}(t_0) = 0$$

or $\frac{dg}{dt}(t_0) = 0$

$\vdots$

$$\alpha_1 \frac{d^{r-1}\varphi_1}{dt^{r-1}}(t_0) + \cdots + \alpha_r \frac{d^{r-1}\varphi_r}{dt^{r-1}} = 0$$

or $\frac{d^{r-1}g}{dt^{r-1}}(t_0) = 0$.

g is a solution satisfying the initial conditions

$$g(t_0) = 0, \quad \frac{dg}{dt}(t_0) = 0, \quad \ldots, \quad \frac{d^{r-1}g}{dt^{r-1}}(t_0) = 0.$$

By the Uniqueness part of the Existence/Uniqueness theorem, g is the zero function (it satisfies the same equation and initial conditions).

Therefore,

$$\alpha_1 \varphi_1 + \cdots + \alpha_r \varphi_r = g = 0$$

with not all α_i equal to zero.

This shows that $\varphi_1, \ldots, \varphi_r$ are linearly dependent. $\square$